Acoustics in lined ducts with sheared mean flow, with applications for aircraft noise

Sjoerd Rienstra & Martien Oppeneer

with major contributions from

Pieter Sijtsma, Bob Mattheij, Werner Lazeroms

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- Acoustic modes in flow ducts, with radial mean flow and temperature gradients.

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Published in: AIAA-2011-2871, AIAA-2013-2172.

Outline

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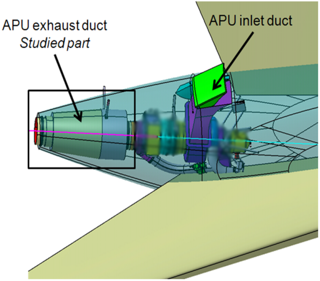
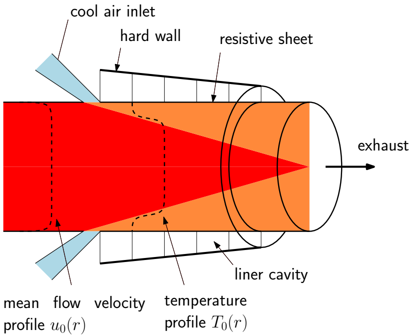
- APU: Auxiliary Power Unit
  - produces power when main engines are switched off
  - to start main engines, AC, ...
  - major source of ramp noise



APU on an Airbus A380

Study sound propagation & attenuation in APU exhaust duct.

Modelling assumptions



- straight, circular, hollow exhaust duct

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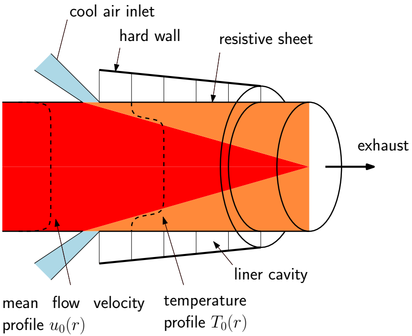

- straight, circular, hollow exhaust duct
- non-uniform parallel mean flow ~~(axially varying)~~

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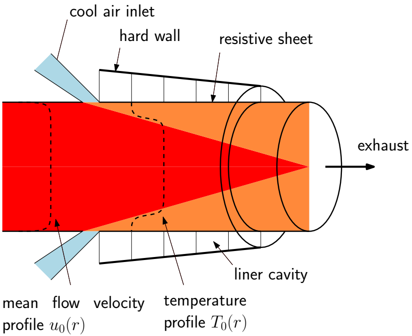

- straight, circular, hollow exhaust duct
- non-uniform parallel mean flow ~~(axially varying)~~
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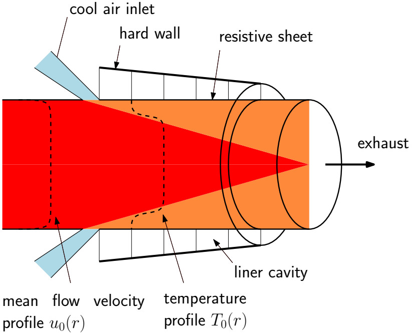

- straight, circular, hollow exhaust duct
- non-uniform parallel mean flow ~~(axially varying)~~
- strong temperature gradients ~~(axially varying)~~
- segmented liner $\Rightarrow$ slowly varying *or* mode-matching
- Euler eqn. & perfect gas: $p = \rho \mathcal{R} T$, $c^2 = \gamma \mathcal{R} T$

Outline

Ordinary & Generalised Pridmore-Brown equation

For perturbations $p_1, \rho_1, \boldsymbol{v}_1$ of a parallel mean flow

$$\boldsymbol{v}_0 = u_0(y,z)\boldsymbol{e}_x, \quad \rho_0 = \rho_0(y,z), \quad c_0 = c_0(y,z), \quad p_0 = \text{const}$$

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the Linearised Euler equations can be reduced to:

Generalised Pridmore-Brown equation (arbitrary cross-section)

For modes of the form $p_1(x,y,z,t) = P(y,z)\,\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t}$:

$$\nabla\cdot\Big(\frac{c_0^2}{\Omega^2}\nabla P\Big) + \Big(1 - \frac{k^2 c_0^2}{\Omega^2}\Big)P = 0, \quad \Omega = \omega - k u_0$$

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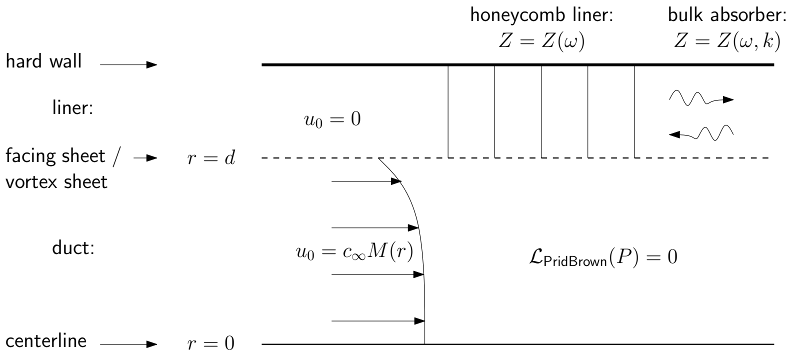
Ordinary Pridmore-Brown equation (circular cross-section)

For $u_0(r), \rho_0(r), c_0(r)$ and $p_1(x,r,\theta,t) = P(r)\,\mathrm{e}^{\mathrm{i}kx - \mathrm{i}\omega t + \mathrm{i}m\theta}$:

$$P'' + \left(\frac{1}{r} + 2\frac{c_0'}{c_0} + 2\frac{k u_0'}{\Omega}\right)P' + \left(\frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2}\right)P = 0$$

Boundary conditions



Ingard-Myers boundary condition for slipping flow:

$$-\mathrm{i}\omega(\boldsymbol{v}_1 \cdot \boldsymbol{n})Z = (-\mathrm{i}\omega + \boldsymbol{v}_0 \cdot \nabla)p_1$$

Boundary value problem

Pridmore-Brown equation

$$P'' + \left(\frac{1}{r} + 2\frac{c_0'}{c_0} + 2\frac{k u_0'}{\Omega}\right) P' + \left(\frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2}\right) P = 0$$

$+$

Boundary conditions

$$\mathrm{i}\omega Z P' = -\rho_0 \Omega^2 P \ \text{ at } r = d, \qquad P \ \text{ is regular at } r = 0$$

$=$

Eigenvalue Problem in k

Countable set of modal solutions: $P_{m\mu}(r)\,\mathrm{e}^{\mathrm{i}k_{m\mu}x - \mathrm{i}\omega t + \mathrm{i}m\theta}$

- eigenfunctions: $P_{m\mu}(r)$
- eigenvalue (modal axial wavenumber): $k_{m\mu}$

Boundary value problem

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$$P'' + \left(\frac{1}{r} + 2\frac{c_0'}{c_0} + 2\frac{ku_0'}{\Omega}\right)P' + \left(\frac{\Omega^2}{c_0^2} - k^2 - \frac{m^2}{r^2}\right)P = 0$$

$$+$$

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Eigenvalue Problem in k

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Non-uniform parallel flow: modes are found numerically

Numerical solution: COLNEW

Write eigenvalue problem as boundary value problem:

- Add k to solution vector by adding equation $k' = 0$
- Fix $P(r)$ at $r = d$

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COLNEW (NL-BVP software package available on netlib):

- Collocation at Gaussian points
- Runge-Kutta monomial basis representation
- Automatic meshing
- Damped Newton solver

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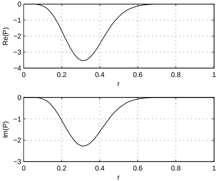
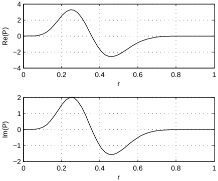
Path-following/predictor-corrector/automatic step-size strategy from known solution to desired solution.

Numerical approach: example of path-following

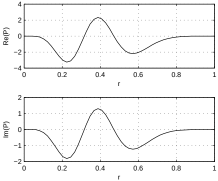
Example of path-following in Z



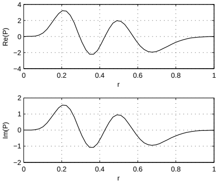
(a) $\mu = 1$



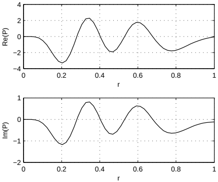
(b) $\mu = 2$



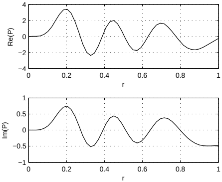
(c) $\mu = 3$



(d) $\mu = 4$



(e) $\mu = 5$



(f) $\mu = 6$

Eigenfunctions for upstream-running modes, $\omega d/c_\infty = 25$, $m = 5$, $Z/\rho_\infty c_\infty = 2 - \mathrm{i}$, $u_0/c_\infty = \frac{2}{3}(1 - \frac{1}{2}r^2)$, uniform temperature.

Numerical results: further tests

Test case borrowed from quantum-mech. potential well problem:

- Pridmore-Brown equation:

$$P'' + \beta(r,k)P' + \gamma(r,k)P = 0$$

- *Quantisation condition* based on high-freq. approximation

$$\int_{r_1}^{r_2} \sqrt{\gamma(r,k)}\,\mathrm{d}r = (n - \tfrac{1}{2})\pi, \quad n = 1, 2, \ldots$$

μ	k_{QC}	k
1	-60.470038	-60.4392
2	-55.761464	-55.7281
3	-51.134207	-51.0980 - 0.0000i
4	-46.605323	-46.5659 - 0.0003i
5	-42.195790	-42.1422 - 0.0212i
6	-37.931052	-37.5622 - 0.3254i

k's for upstream-running modes.

- High-freq. approx. & numerical result: excellent agreement

Outline

Options for varying Z

General solution by sum over modes

$$p_m(r,x) = \sum_{\mu=1}^{\mu^{\mathsf{max}}} \left[A^{+}_{m\mu} P^{+}_{m\mu}(r)\, \mathrm{e}^{\mathrm{i}k^{+}_{m\mu}x} + A^{-}_{m\mu} P^{-}_{m\mu}(r)\, \mathrm{e}^{\mathrm{i}k^{-}_{m\mu}x} \right]$$

Classic option for (piecewise) varying Z is *Mode Matching*.

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This is efficient and well-established (BAHAMAS◄NLR) for no-flow and uniform flow conditions, mainly because

- exact solutions of PB equation ($P_{m\mu} = J_m$ Bessel functions),
- exact modal inner products (integrals) at interfaces.

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Questions for non-uniform mean flow:

- What can we do with a *slowly* varying impedance?
- Can we improve the efficiency of the mode-matching?

Outline

Slowly varying modes:

Assumptions

- $Z(x)$ has an inherent length scale $L \gg d$, no sudden changes. We rewrite

$$Z := Z(\varepsilon x) = Z(X), \quad \varepsilon = \frac{d}{L} \ll 1. \quad X = \varepsilon x.$$

- No modal interaction (reflection, cut-on/cut-off, etc)
- Mode, slowly varying in axial direction (WKB Ansatz)

$$\tilde{p}_m(r, X) = P(r, X) \exp\left(\frac{\mathrm{i}}{\varepsilon} \int_0^X \kappa(\eta) \mathrm{d}\eta\right)$$

Eigenfunction $P(r, X)$ and wave number $\kappa(X)$ to be found.

WKB for slowly varying Z

- Expand in ε

$$P(r,X) = P_0(r,X) + \varepsilon P_1(r,X) + O(\varepsilon^2)$$

- To leading order, the slowly varying mode of order m, μ

$$P_0(r,X) = N(X)\psi_{m\mu}(r,X), \quad \text{with} \quad \kappa = \kappa_{m\mu}(X)$$

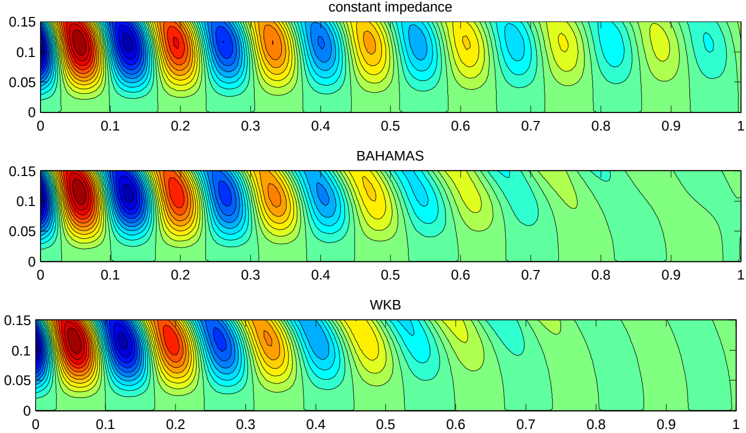
  where $\psi_{m\mu}(r,X)$ and $\kappa_{m\mu}(X)$ are modal solutions per X

- $N(X)$ is found from solvability condition for P_1, eventually leading to

$$N(X)^2 = N_0^2 \exp\left(-\int_0^X \frac{f(\eta)}{g(\eta)}\mathrm{d}\eta\right)$$

  where $f(X), g(X)$ are complicated but explicit functions of X, ω, u_0, ρ_0, c_0, $Z(X)$, $\psi_{m\mu}$, and $\kappa_{m\mu}$.

Linear $Z(X)$



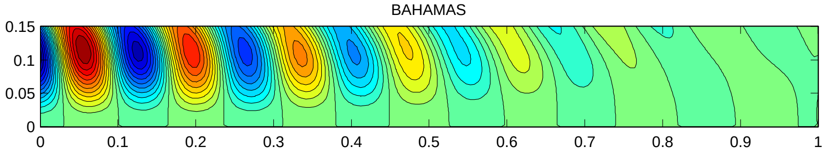
$Z/\rho_\infty c_\infty$ varies linearly from $1.5 - \mathrm{i}$ to $1.5 + \mathrm{i}$. BAHAMAS: 10 segments.

$\Rightarrow$ x-dependency of Z is important

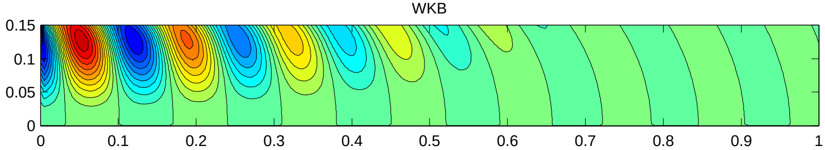
$\Rightarrow$ BAHAMAS and WKB agree well ($\varepsilon \approx 0.2$)

Numerical results: non-uniform flow velocity

Non-uniform velocity



(a) Uniform mean flow velocity with $u_0/c_\infty = 0.3$.

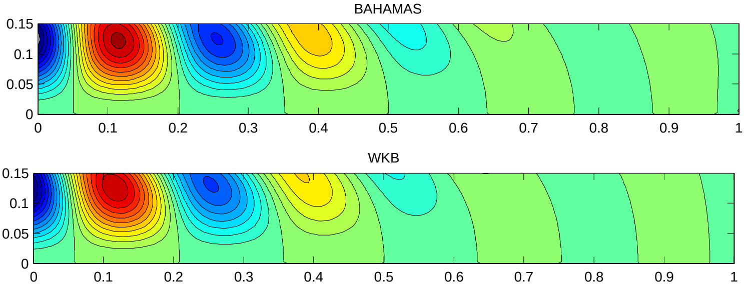


(b) $u_0/c_\infty = 0.3 \cdot \frac{4}{3}(1 - \frac{1}{2}r^2)$

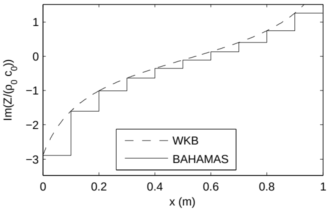
$\omega d/c_\infty = 10$, $m = 2$, $\mu = 1$, $Z/\rho_\infty c_\infty$ varies linearly from $1.5 - \mathrm{i}$ to $1.5 + \mathrm{i}$ so $\varepsilon \approx 0.2$.

$\Rightarrow$ Non-uniformity of mean flow velocity is important

Helmholtz resonator (no resonance)

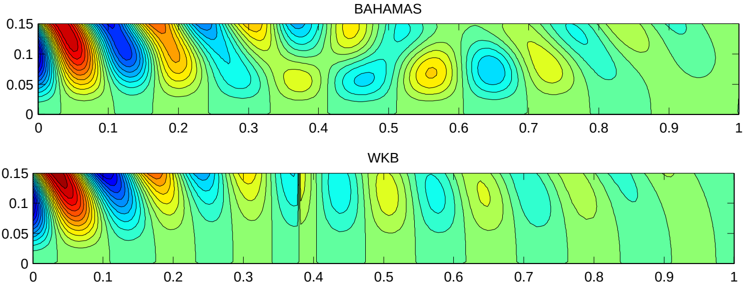


$\omega d/c_\infty = 6$, $m = 2$, $\mu = 1$, uniform mean flow velocity $u_0/c_\infty = 0.3$

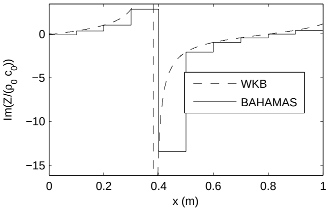


$\Rightarrow$ No resonance and $\varepsilon \approx 0.3$: BAHAMAS and WKB show good agreement

Helmholtz resonator (passing resonance)

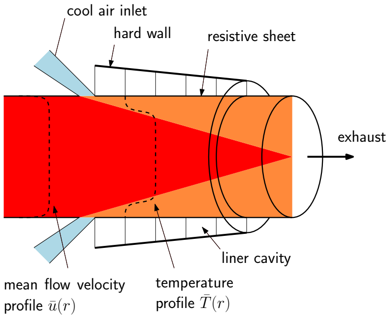


$\omega d/c_\infty = 10$, $m = 2$, $\mu = 1$, uniform mean flow velocity $u_0/c_\infty = 0.3$.

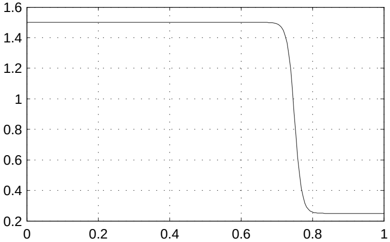


$\Rightarrow$ Resonance: WKB assumptions not valid ($Z(x)$ not slowly varying, intermodal scattering)

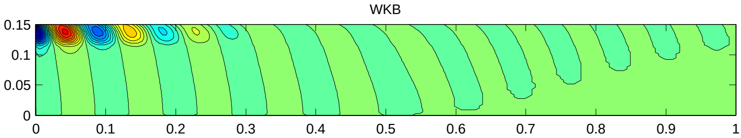
Realistic APU exhaust: strong temperature gradient



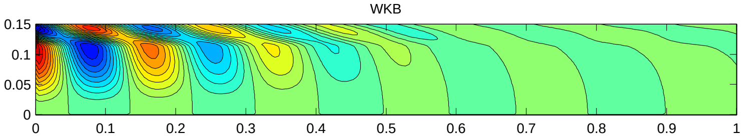
(a) APU exhaust duct geometry with cool air inlet.



(b) Temperature profile $T/T_\infty = \frac{1}{4} + \frac{5}{8}\big(1 + \tanh\big(50(\frac{3}{4} - r)\big)\big)$.



(a) WKB, $\mu = 1$.



(b) WKB, $\mu = 2$.

$\omega d/c_\infty = 10$, $m = 2$, uniform velocity $u_0/c_\infty = 0.3$,
$Z(x)/\rho_\infty c_\infty$ linear: $1.5 - \mathrm{i}$ to $1.5 + \mathrm{i}$.

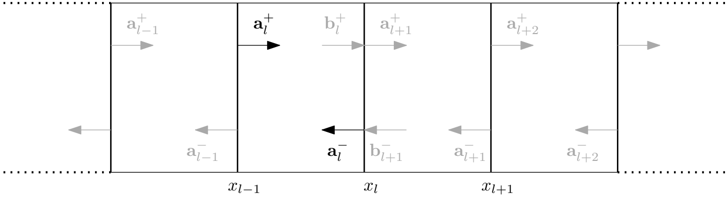
$\Rightarrow$ 2 different sound speeds: *2 concentric ducts*

Sound refracts from warm to cold

Enhances effect of lining

Outline

Mode-Matching Basics

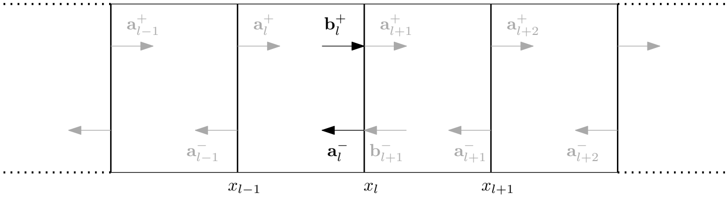


Total field in segment l: sum of left- and right-running waves

$$p_l(x,r) = \sum_{\mu=1}^{\infty} \left(a_{l,\mu}^{+} P_{l,\mu}^{+}(r)\, \mathrm{e}^{\mathrm{i}k_{l,\mu}^{+}(x-x_{l-1})} + a_{l,\mu}^{-} P_{l,\mu}^{-}(r)\, \mathrm{e}^{\mathrm{i}k_{l,\mu}^{-}(x-x_l)} \right)$$

(same for velocity)

Mode-Matching Basics

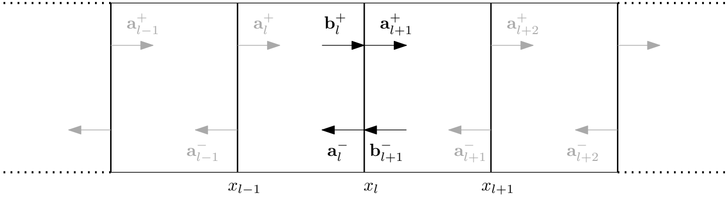


At the interface at $x = x_l$:

$$p_l(r) = \sum_{\mu=1}^{\mu^{\max}} \Big(b^+_{l,\mu} P^+_{l,\mu}(r) + a^-_{l,\mu} P^-_{l,\mu}(r) \Big).$$

(same for velocity)

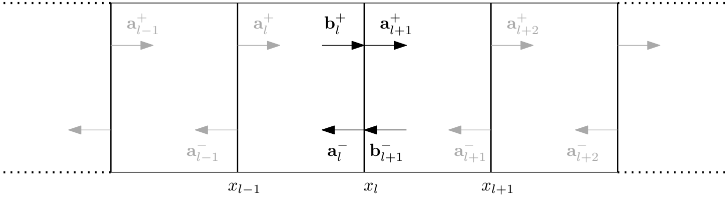
Mode-Matching Basics



Continuity of pressure at $x = x_l$ leads to

$$p_l(x_l, r) = p_{l+1}(x_l, r)$$

Mode-Matching Basics

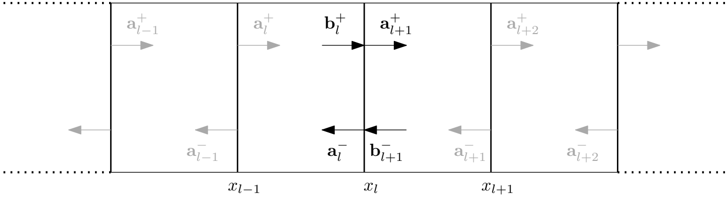


Continuity of pressure at $x = x_l$ leads to

$$\sum_{\mu=1}^{\mu^{\max}} \left(b^{+}_{l,\mu} \; P^{+}_{l,\mu} \qquad + a^{-}_{l,\mu} \; P^{-}_{l,\mu} \qquad \right)$$

$$= \sum_{\mu=1}^{\mu^{\max}} \left(a^{+}_{l+1,\mu} \; P^{+}_{l+1,\mu} \qquad + b^{-}_{l+1,\mu} \; P^{-}_{l+1,\mu} \qquad \right)$$

Mode-Matching Basics

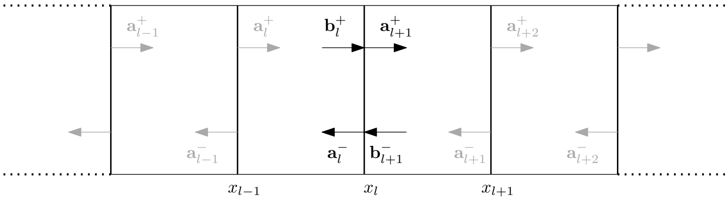


inner products with suitable test functions Ψ_ν, e.g. $= J_m(\alpha_\nu r)$

$$\sum_{\mu=1}^{\mu^{\text{max}}} \Big(b_{l,\mu}^{+} (P_{l,\mu}^{+}, \Psi_\nu) + a_{l,\mu}^{-} (P_{l,\mu}^{-}, \Psi_\nu) \Big)$$

$$= \sum_{\mu=1}^{\mu^{\text{max}}} \Big(a_{l+1,\mu}^{+} (P_{l+1,\mu}^{+}, \Psi_\nu) + b_{l+1,\mu}^{-} (P_{l+1,\mu}^{-}, \Psi_\nu) \Big)$$

Mode-Matching Basics



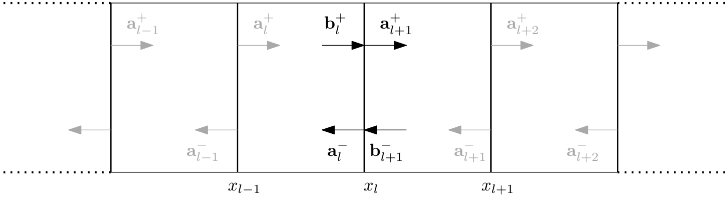
inner products with suitable test functions Ψ_ν, e.g. $= J_m(\alpha_\nu r)$

$$\sum_{\mu=1}^{\mu^{\mathsf{max}}} \Big(b^+_{l,\mu} (P^+_{l,\mu}, \Psi_\nu) + a^-_{l,\mu} (P^-_{l,\mu}, \Psi_\nu) \Big)$$

$$= \sum_{\mu=1}^{\mu^{\mathsf{max}}} \Big(a^+_{l+1,\mu} (P^+_{l+1,\mu}, \Psi_\nu) + b^-_{l+1,\mu} (P^-_{l+1,\mu}, \Psi_\nu) \Big)$$

Similar for continuity of axial velocity.

Mode-Matching Basics



Results in linear system to be solved

$$\begin{bmatrix} \mathbf{A}^+ & \mathbf{A}^- \\ \mathbf{C}^+ & \mathbf{C}^- \end{bmatrix} \begin{bmatrix} \mathbf{b}_l^+ \\ \mathbf{a}_l^- \end{bmatrix} = \begin{bmatrix} \mathbf{B}^+ & \mathbf{B}^- \\ \mathbf{D}^+ & \mathbf{D}^- \end{bmatrix} \begin{bmatrix} \mathbf{a}_{l+1}^+ \\ \mathbf{b}_{l+1}^- \end{bmatrix}.$$

Computing inner products

Matrix entries are inner products

$$A^{\pm}_{\nu\mu} = (P^{\pm}_{l,\mu}, \Psi_\nu) = \int_0^d P^{\pm}_{l,\mu}(r)\Psi_\nu(r) r \,\mathrm{d}r$$

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- All inner-products have to be determined at all interfaces by quadrature
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Summary of new matching method

Classical	$\rightarrow$	**new**	mode-matching
(P_μ, Ψ_ν)	$\rightarrow$	$\langle \boldsymbol{F}_\mu, \boldsymbol{\Psi}_\nu \rangle$	

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Classical (CMM) $\rightarrow$ **new (BLM)** mode-matching

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expensive less accurate	$\rightarrow$	cheap accurate

Closed form integrals of 2D eigenmodes

Prototype example of *Generalised Prid-Brown* : Helmholtz eqn

$$\nabla^2\psi + \beta^2\psi \quad = 0$$

on arbitrarily shaped cross-section $\mathcal{A}$

Closed form integrals of 2D eigenmodes

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2D inner-product for Helmholtz eigenfunctions

$$\langle\!\langle \phi, \psi \rangle\!\rangle = \frac{1}{\alpha^2 - \beta^2}\int_{\Gamma}(\phi\nabla\psi\cdot\boldsymbol{n} - \psi\nabla\phi\cdot\boldsymbol{n})\mathrm{d}\ell,$$

for arbitrary boundary conditions on ϕ and ψ

What if $\alpha = \beta$ and $\phi = \psi$? Something similar.

- Circular duct: Helmholtz equation → Bessel equation
- Substitute into 2D inner-product:

$$\phi = J_m(\alpha r)\,\mathrm{e}^{\mathrm{i}m\theta},\ \psi = J_m(\beta r)\,\mathrm{e}^{-\mathrm{i}m\theta}$$

Closed form integrals of 1D eigenmodes

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- Substitute into 2D inner-product:

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1D inner-product of Bessel functions

$$\begin{aligned}\langle J_m(\alpha r), J_m(\beta r)\rangle &= \int_0^1 J_m(\alpha r) J_m(\beta r)\, r\mathrm{d}r \\ &= \frac{1}{\alpha^2 - \beta^2}\left[\beta J_m(\alpha) J_m'(\beta) - \alpha J_m'(\alpha) J_m(\beta)\right]\end{aligned}$$

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If $\alpha = \beta$: something similar.

Closed form integrals for Generalised P-B modes

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Closed form integral of parallel flow modes

$$\langle\!\langle \boldsymbol{F}, \widetilde{\boldsymbol{F}} \rangle\!\rangle =$$

$$\iint_{\mathcal{A}} \frac{1}{\widetilde{\Omega}} \left[\left(\frac{u_0}{\rho_0 c_0^2} + \frac{\widetilde{k}}{\rho_0 \widetilde{\Omega}} \right) \widetilde{P} P + \frac{\omega}{\widetilde{\Omega}} \widetilde{P} U - \rho_0 u_0 (\widetilde{V} V + \widetilde{W} W) \right] \mathrm{d}S$$

$$= \frac{\mathrm{i}}{k - \widetilde{k}} \int_{\Gamma} \frac{\widetilde{P}(V n_y + W n_z) - (\widetilde{V} n_y + \widetilde{W} n_z) P}{\widetilde{\Omega}} \, \mathrm{d}\ell,$$

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Something similar for $k = \widetilde{k}$.

Substitute for circular symmetric geometry. . .

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modes of the form $\boldsymbol{F}(r)\,\mathrm{e}^{\pm\mathrm{i}m\theta}$

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Closed form integrals for radial Pridmore-Brown modes

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Exact integrals of radial Pridmore-Brown modes

$$\langle \boldsymbol{F}, \widetilde{\boldsymbol{F}} \rangle =$$

$$\int_0^d \frac{1}{\widetilde{\Omega}} \left[\left(\frac{u_0}{\rho_0 c_0^2} + \frac{\widetilde{k}}{\rho_0 \widetilde{\Omega}} \right) P\widetilde{P} + \frac{\omega}{\widetilde{\Omega}} U\widetilde{P} - \rho_0 u_0 (V\widetilde{V} + W\widetilde{W}) \right] r\,\mathrm{d}r$$

$$= \frac{\mathrm{i}d}{k - \widetilde{k}} \left[\frac{\widetilde{P}V - \widetilde{V}P}{\widetilde{\Omega}} \right]_{r=d}$$

Weighted products of Pridmore-Brown eigenfunctions.
Something similar for $k = \widetilde{k}$.

Some special cases

With Ingard-Myers condition (slipping flow)

$$\langle \boldsymbol{F}, \widetilde{\boldsymbol{F}} \rangle = \left[\frac{\mathrm{i} d \widetilde{P} P}{(k - \widetilde{k}) \widetilde{\Omega} \omega} \left(\frac{\Omega}{Z} - \frac{\widetilde{\Omega}}{\widetilde{Z}} \right) \right]_{r=d}$$

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For hard walls, or same impedance $Z = \widetilde{Z}$:

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Bilinear map-based mode-matching

Classic mode-matching (CMM)

$$\sum_{\mu=1}^{\mu_l} b^+_{l,\mu}(P^+_{l,\mu}, \Psi_\nu) + a^-_{l,\mu}(P^-_{l,\mu}, \Psi_\nu) \\ = \sum_{\mu=1}^{\mu_{l+1}} a^+_{l+1,\mu}(P^+_{l+1,\mu}, \Psi_\nu) + b^-_{l+1,\mu}(P^-_{l+1,\mu}, \Psi_\nu)$$

(same for velocity) with test functions (for example)

$$\Psi_\nu = J_m(\alpha_\nu r)$$

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Quadrature required for (P_μ, Ψ_ν) terms (non-uniform flow)

Bilinear map-based (BLM) mode-matching

$$\sum_{\mu=1}^{\mu_l} b^+_{l,\mu}\langle \boldsymbol{F}^+_{l,\mu}, \boldsymbol{\Psi}_\nu\rangle + a^-_{l,\mu}\langle \boldsymbol{F}^-_{l,\mu}, \boldsymbol{\Psi}_\nu\rangle$$
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but now as test functions the same modes:

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but now as test functions the same modes:

$$\boldsymbol{\Psi}_\nu = \boldsymbol{F}_{l,\nu}$$

No extra calculations *and* $\langle \boldsymbol{F}_\mu, \boldsymbol{\Psi}_\nu\rangle$ in closed form

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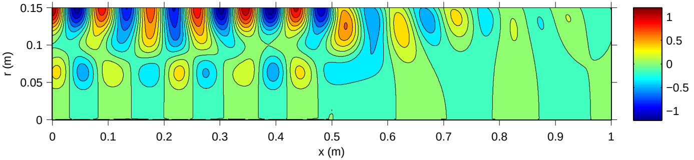
Comparing CMM and BLM

Test configurations

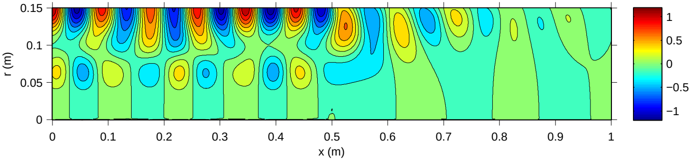
- Length: 1m
- Radius: 15cm
- hard wall – soft wall, interface at $x = 0.5$m
- $\mu^{\text{max}} = 50$ modes in both directions

Configuration	I	II	III
Helmholtz & m	$\omega d/c_\infty = 13.86,\ m = 5$	$\omega d/c_\infty = 8.86,\ m = 5$	$\omega d/c_\infty = 15,\ m = 5$
Temperature	$T_0/T_\infty = 1$	$T_0/T_\infty = 1$	$T_0/T_\infty = 2\log(2)(1 - \frac{r^2}{2})$
Mean flow	$u_0/c_\infty = 0.5 \cdot (1 - r^2)$	$u_0/c_\infty = 0.3 \cdot \frac{4}{3}(1 - \frac{r^2}{2})$	$u_0/c_\infty = 0.3 \cdot \tanh(10(1 - r))$
Impedance	$Z/\rho_\infty c_\infty = 1 - \mathrm{i}$	$Z/\rho_\infty c_\infty = 1 + \mathrm{i}$	$Z/\rho_\infty c_\infty = 1 - \mathrm{i}$
Incident mode	$\mu = 1$	$\mu = 1$	$\mu = 2$

Real part of pressure



(a) Classical mode-matching.

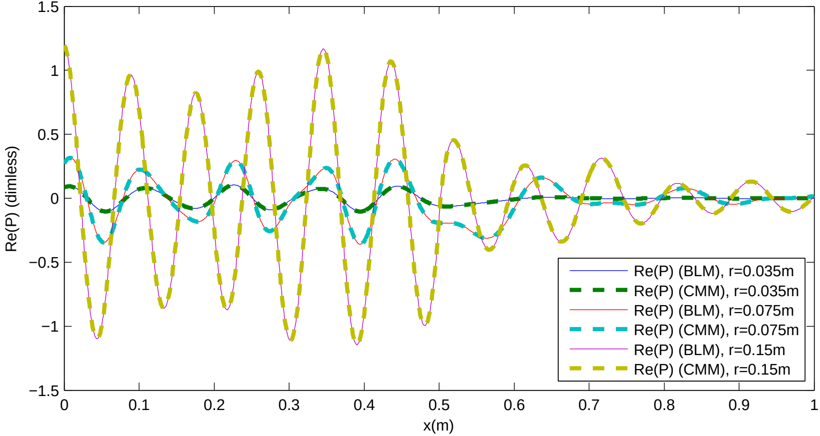


(b) Bilinear map-based mode-matching.

Perfect match between BLM and CMM results

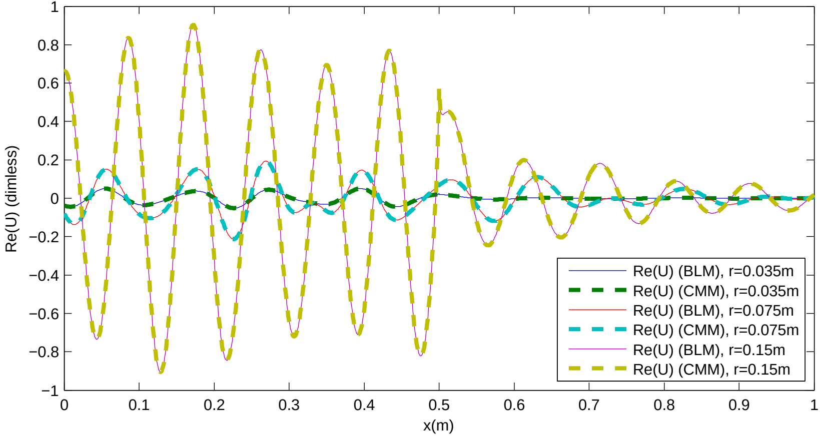
Numerical results — Conf I: no-slip flow, uniform temp

Pressure at $r = \{0.035, 0.075, 0.15\}\,\mathrm{m}.$



Perfect match between BLM and CMM results

Axial velocity at $r = \{0.035, 0.075, 0.15\}\,\mathrm{m}.$



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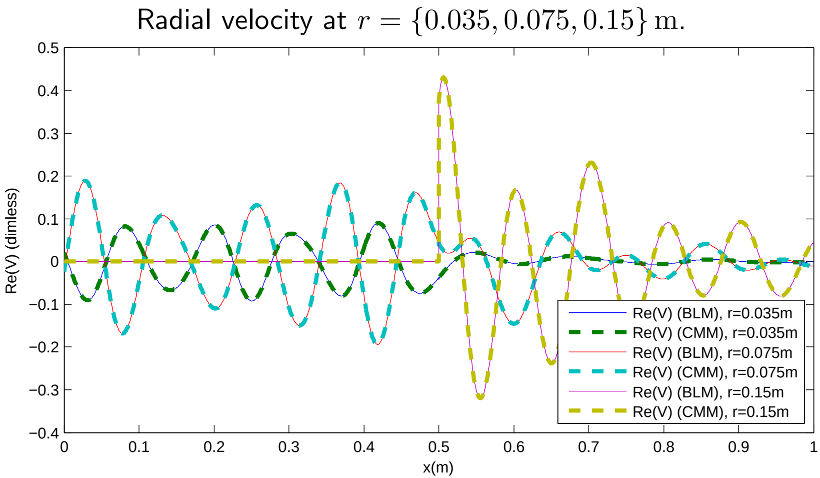
Numerical results — Conf I: no-slip flow, uniform temp



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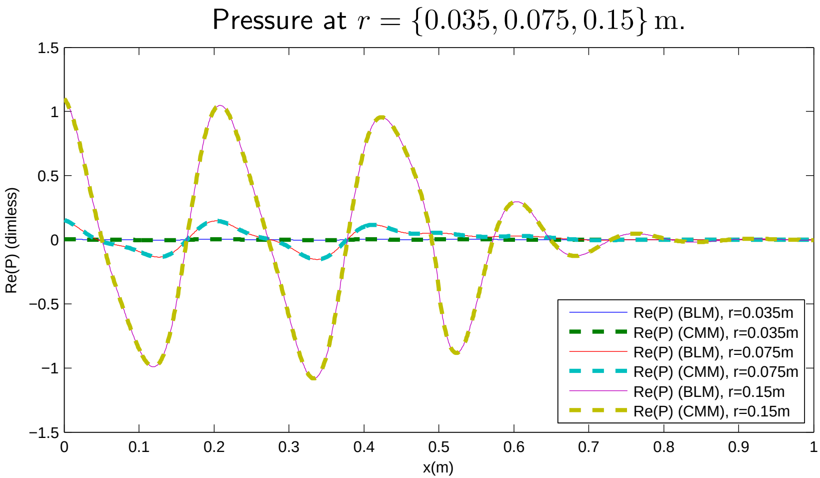

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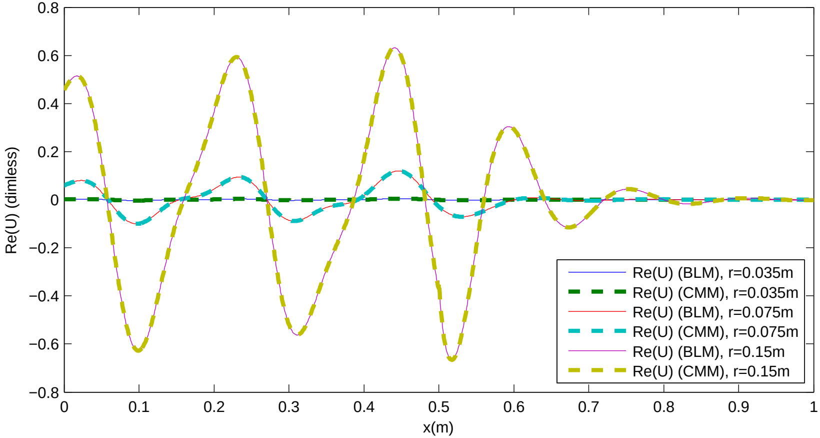

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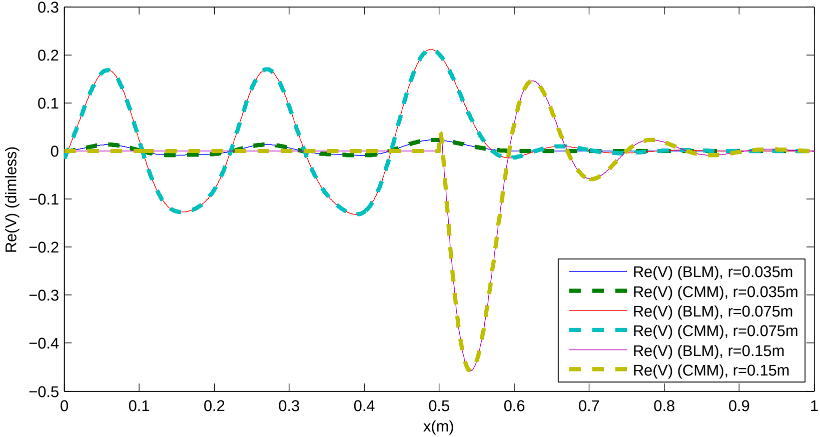

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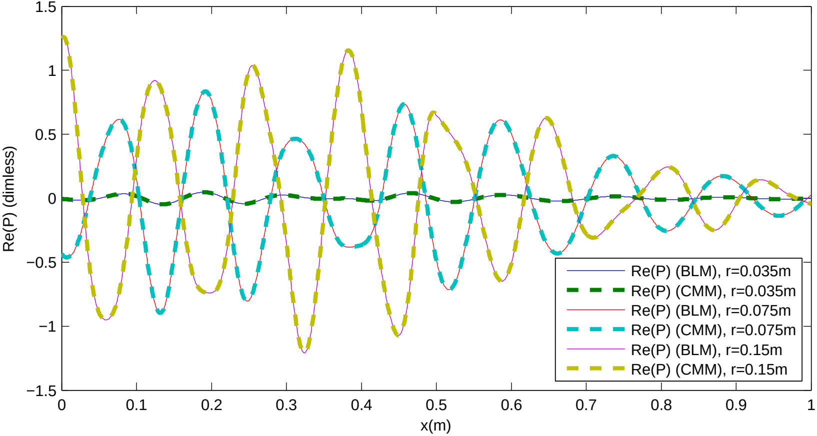
Pressure at $r = \{0.035, 0.075, 0.15\}\,\mathrm{m}.$



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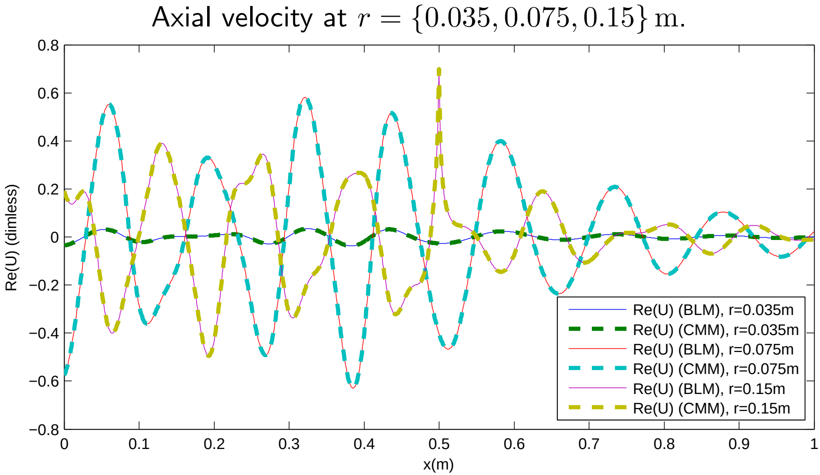

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Numerical results — Conf III: bndary layer, non-unif. temp

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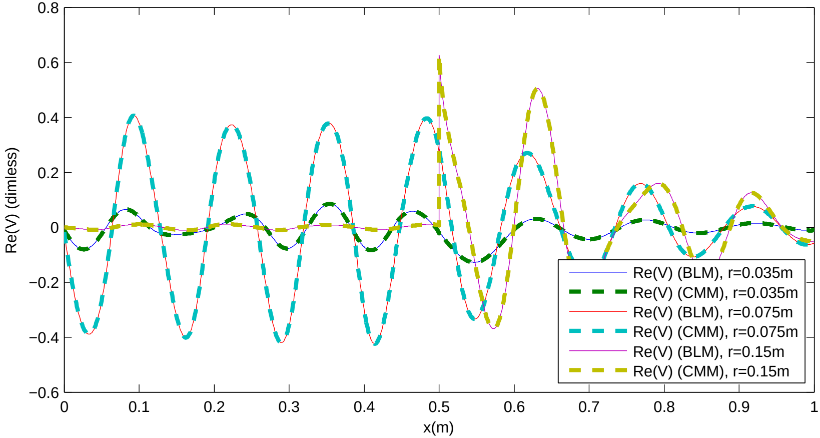

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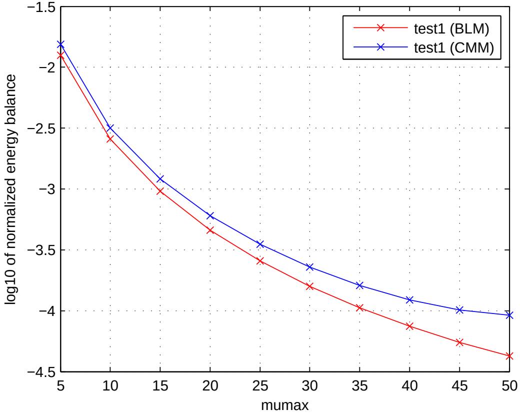
Numerical results — Energy balance

Energy balance (Myers' Energy Corollary) vs $\mu^{\max}$ for conf. I



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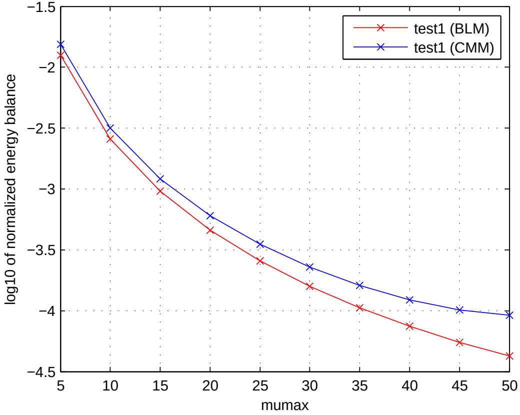

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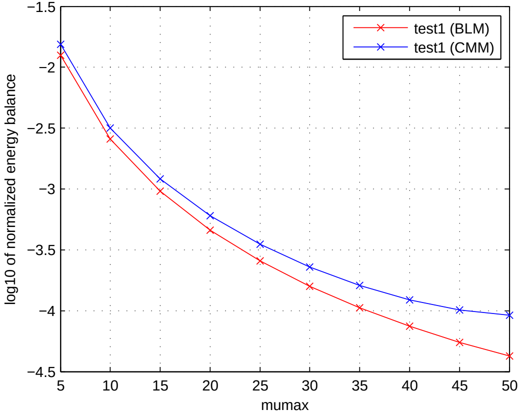

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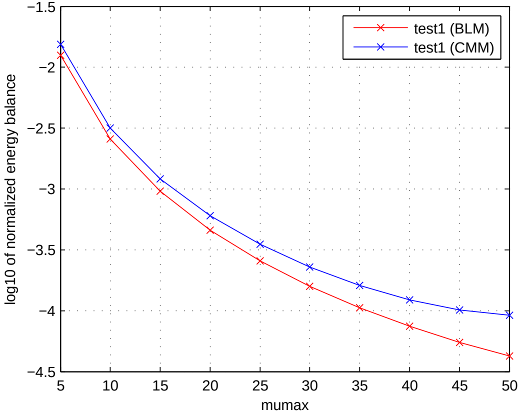

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It can be shown that:

$$p < -1 \Rightarrow \text{uniform convergence of modal series}$$
$$\Rightarrow \text{edge condition satisfied}$$

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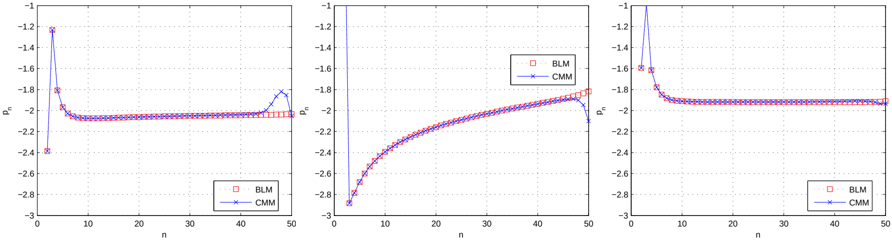

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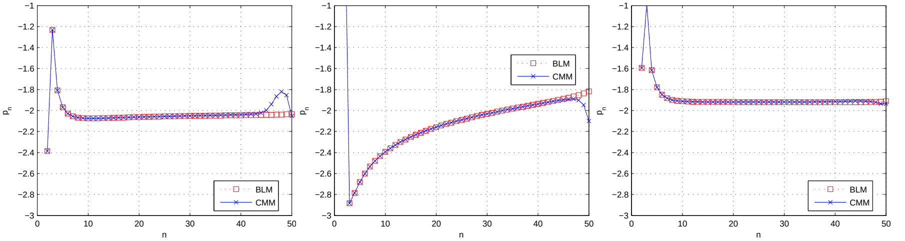

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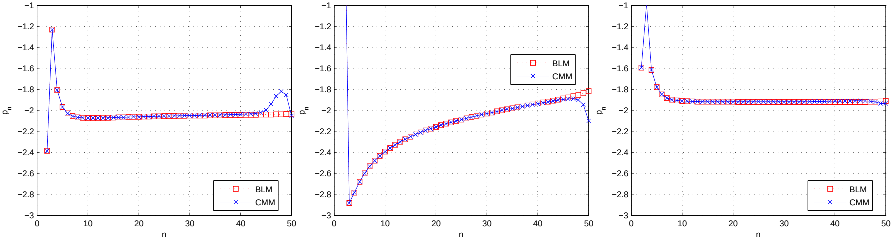

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Convergence of p_n reveals inaccuracies of CMM amplitudes:

BLM amplitudes smoother than CMM as $n \to \infty$: no quadrature inaccuracies for BLM. Explains energy behaviour.

Outline

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- Using standard BVP solver COLNEW
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Slowly varying mode-approximation applied to typical APU duct

- Small enough ε: favourable comparison with BAHAMAS (mode matching)
- WKB fails when Helmholtz liner passes resonance
- Strong effects of temperature and mean flow gradients.
- The need for Mode Matching was clear

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Bilinear map-based mode-matching (BLM):

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  - Solutions in very good agreement with CMM
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- A fine task in functional analysis remains ...