A combustion instability model accounting for dynamic flame-flow-acoustic interactions

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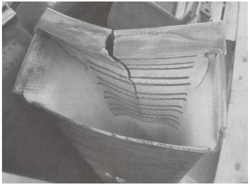
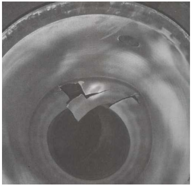
1 Introduction

- **Applications:**
  - Jet engines
  - Rocket engines
  - Gas turbine engines
- **Problems:**
  - Vibrations
  - Structural fatigue
  - Increase in fuel consumption


Phenomena

- Self-sustained, large amplitude pressure fluctuations and flame oscillations
- Generally occurs around the characteristic frequency of the combustor

Objective:

Understanding the fundamental mechanisms of combustion instabilities

Physical mechanisms: Flame-Flow-Acoustic interactions

1. Flame excites the acoustic pressure
   - Acoustic pressure amplifies according to Rayleigh's criterion, i.e when heat release and acoustic pressure are in phase.
2. Acoustic velocity and acoustic acceleration advect and modulate the flame front.
3. The flame influences the hydrodynamic flow, which in turn influences the flame.

Multi-scale, Extremely challenging for DNS!

Existing models and comparison

The G-equation [see e.g. Dowling, JFM, 1999]

- Model the advection effect of acoustic on the flame
- Purely kinematic (no hydrodynamics)
- Ignores dynamic effect of acoustics

Hydrodynamic theory of flames

- Flame-flow interaction model, no acoustic considerations
  [Pelcé & Clavin, JFM 1982], [Matalon & Matkowsky, JFM 1982]
- Reduces to Michelson-Sivashinsky equation (M-S)

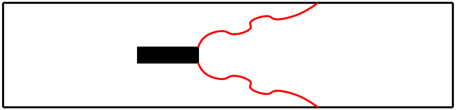
	Advection	Hydrodynamics (D-L)	Flame-acoustics coupling	R-T
G-equation	✓	✗	✓	✗
M-S	✓	✓	✗	✗
Aim:	✓	✓	✓	✓

G-equation

$$\frac{\partial G}{\partial t} = -\boldsymbol{u} \cdot \nabla G + S_u |\nabla G|, \qquad \frac{\partial F}{\partial t} = u_G - S_u \sqrt{1 + \left(\frac{\partial F}{\partial y} \right)^2}$$

- $G = G(x, y, t)$: level set function representing the flame
- S_u: normal flame speed propagation
- Write $G(x, y, t) = x - F(y, t)$ and consider 1D velocity fluctuations
- $u_G = u_G(t)$: "u gutter", acoustic velocity at the flame (1D)

- Allows for wrinkling of anchored flames...



- ... but not for freely propagating flames.

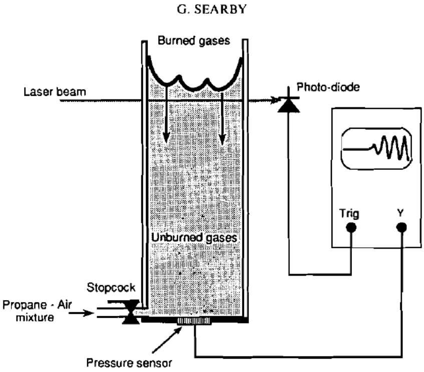
The M-S equation (2D)

$$\begin{cases} \frac{\partial \varphi}{\partial t} = \frac{1}{2} I(\varphi; y) + \frac{1}{\gamma} \frac{\partial^2 \varphi}{\partial y^2} + \frac{1}{2} \left(\frac{\partial \varphi}{\partial y} \right)^2, \\ \widehat{I(\varphi; y)}(k, t) = |k| \widehat{\varphi}(k, t), \end{cases}$$

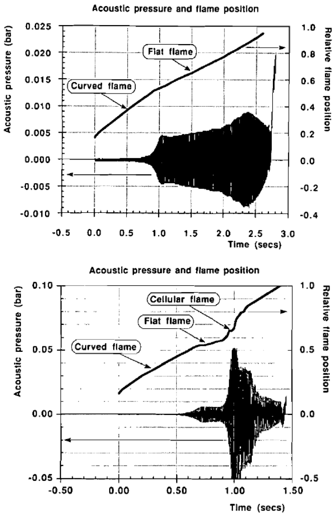
- φ : flame shape
- γ : "free" parameter
- $\widehat{\ }$: Fourier transform in y direction

- Good for unconfined freely propagating flames
- Predicts Darrieus-Landau instability: i.e. flames tend to curve

Setup



Results



Flame-flow-acoustics in the literature

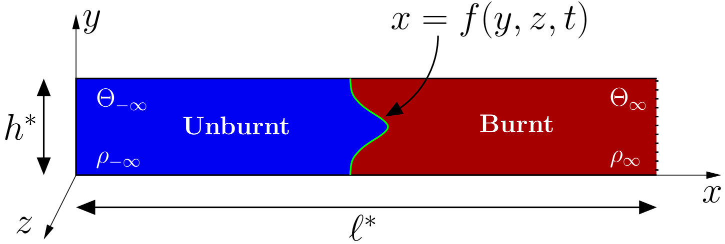
Externally imposed acoustics

- For example [Markstein & Squire, 1955], [Searby & Rochwerger, JFM 1991], [Clanet & Searby, PRL 1998], [Bychkov, PoF 1999]
- No consideration of back-action of the flame onto the acoustics
- Focused on secondary instability

Including spontaneous acoustic field

- [Pelcé & Rochwerger, JFM 1992]
- First mathematical treatment of flow-flame-acoustics interactions
- Ad-hoc modelling of flame profile as cosine function
- Focused on primary instability

- Geometry



Physical assumptions

- One-step irreversible chemical reaction
- Fuel deficient reactant: lean combustion
- Mixture obey state equation for perfect gas
- Newtonian compressible fluid

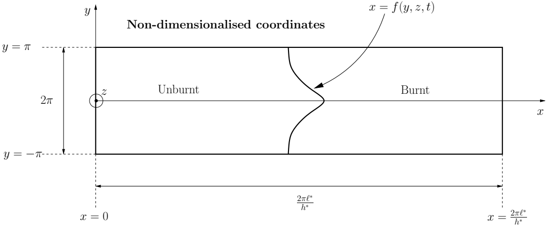
Equations to be solved

Equations

- Conservation of mass
- Conservation of momentum
- Transport equation governing the diffusion of chemical species
- Energy conservation
- State equation

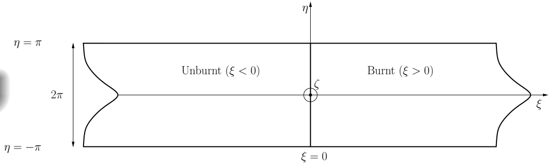
Main variables of non-dimensional problem

- $\boldsymbol{u}$, ρ, p, θ
- M: Mach number
- q: heat release
- β: activation energy
- $\rho_{-\infty}$, $\theta_{-\infty}$,U_L
- δ: flame thickness
- $q = (\theta_{\infty} - \theta_{-\infty})/\theta_{-\infty}$
- $\mathrm{M_a}$: Markstein number



Change of variable $(x,y,z,t) \to (\xi,\eta,\zeta,\tau)$
$\xi = x - f(y,z,t)$, $\eta = y$, $\zeta = z$ and $\tau = t$

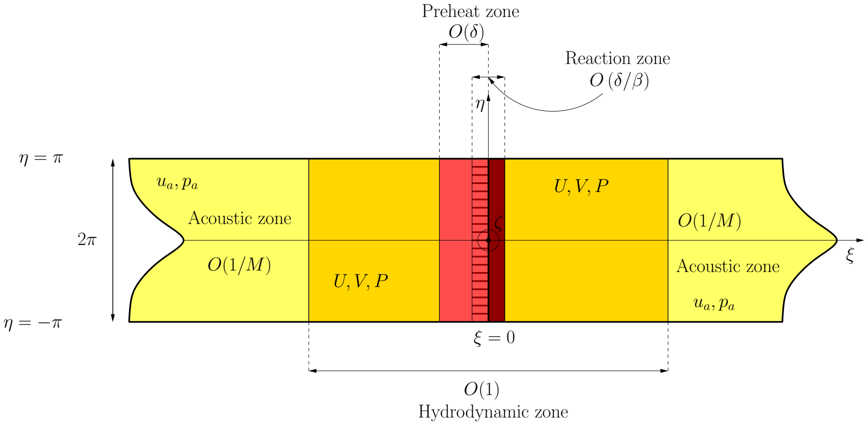
$$\boldsymbol{u} = u\boldsymbol{e_\xi} + \boldsymbol{v}$$



$$\tilde{\nabla} = \begin{pmatrix} \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{pmatrix}$$

2 Asymptotic analysis

Different scalings of the problem



- Large-Activation-Energy : $\beta \gg 1$
- Low Mach Number: $M \ll 1$
- Thin flame : $\delta \ll 1$

$$\frac{\delta}{\beta} \ll \delta \ll 1 \ll \frac{1}{M}$$

[Hydro. theo. flames]

[Wu et al, JFM 2003]

Acoustic zone

Stretch of variable $\widetilde{\xi} = M\xi$

Acoustic Equations

$$\begin{cases} \dfrac{\partial p_a}{\partial \tau} + \dfrac{\partial u_a}{\partial \widetilde{\xi}} &= 0 \\ R\dfrac{\partial u_a}{\partial \tau} + \dfrac{\partial p_a}{\partial \widetilde{\xi}} &= 0 \end{cases}$$

- $\rho = R$ to first order in δ

Acoustic Jumps (weakly nonlinear)

$$\begin{cases} [\![p_a]\!]_-^+ = 0 \\ [\![u_a]\!]_-^+ = \mathcal{J}_a(\tau) = \dfrac{q}{2}\overline{\left(\widetilde{\nabla} F\right)^2} \end{cases}$$

- $f = F$ to first order in δ
- Acoustic-flame coupling

Hydrodynamic zone 1/2: $(u, \boldsymbol{v}, p) = (U, \boldsymbol{V}, P) + O(\delta^2)$

- Linearising the hydrodynamic equations and the jumps to second order in δ leads to:

Hydrodynamic equations

$$\begin{cases} \dfrac{\partial U}{\partial \xi} + \tilde{\nabla} \cdot \boldsymbol{V} &= 0, \\ R\dfrac{\partial U}{\partial \tau} + \dfrac{\partial U}{\partial \xi} &= -\dfrac{\partial P}{\partial \xi} + \delta \, \mathrm{Pr} \, \Delta U, \\ R\dfrac{\partial \boldsymbol{V}}{\partial \tau} + \dfrac{\partial \boldsymbol{V}}{\partial \xi} &= -\tilde{\nabla} P + \delta \, \mathrm{Pr} \, \Delta \boldsymbol{V}, \end{cases}$$

Jump conditions

$$\begin{cases} [U]_-^+ &= \mathcal{J}_U \left(F, \boldsymbol{V}^-\right) \\ [\boldsymbol{V}]_-^+ &= \mathcal{J}_{\boldsymbol{V}} \left(F, \boldsymbol{V}^-\right) \\ [P]_-^+ &= \mathcal{J}_P \left(F, U^-, \boldsymbol{V}^-, \mathcal{B}_a(\tau)\right) \end{cases}$$

$$U^- = U(0^-, \eta, \zeta, \tau)$$

Weakly nonlinear Flame equation

$$\frac{\partial F}{\partial \tau} = U^- - \boldsymbol{V}^- \cdot \tilde{\nabla} F - (\nabla F)^2/2 + \delta \mathsf{M_a} \tilde{\nabla}^2 F + \delta \mathcal{E}_F \left(\boldsymbol{V}^-, F\right)$$

- $\mathcal{B}_a(\tau) = \left[\!\left[\dfrac{\partial p_a}{\partial \tilde{\xi}} \right]\!\right]_-^+ = \left[\!\left[-R \dfrac{\partial u_a}{\partial \tau} \right]\!\right]_-^+$

- Acoustic-flow coupling
- Flow-flame coupling

What now?

- Using Fourier analysis, this whole system can be simplified.

- Making some simplifications, the equations can be reduced to
  - The G-equation
  - The M-S equation

Objective

Retaining key terms to allow for a simple model accounting for the three-way coupling physics of the problem.

Hydrodynamic zone 2/2: $(u, \boldsymbol{v}, p) = (U, \boldsymbol{V}, P) + O(\delta)$

- Considering only the leading order in δ, and partially linearising the flame equation leads to:

Hydrodynamic equations

$$\begin{cases} \dfrac{\partial U}{\partial \xi} + \widetilde{\nabla} \cdot \boldsymbol{V} &= 0 \\ R\dfrac{\partial U}{\partial \tau} + \dfrac{\partial U}{\partial \xi} &= -\dfrac{\partial P}{\partial \xi} \\ R\dfrac{\partial \boldsymbol{V}}{\partial \tau} + \dfrac{\partial \boldsymbol{V}}{\partial \xi} &= -\widetilde{\nabla} P \end{cases}$$

Jump conditions

$$\begin{cases} [U]_-^+ &= 0 \\ [\boldsymbol{V}]_-^+ &= -q\left(\widetilde{\nabla} F\right) \\ [P]_-^+ &= -\left(\mathcal{B}_a(\tau) + \dfrac{qG}{1+q}\right) F \end{cases}$$

Weakly nonlinear Flame equation

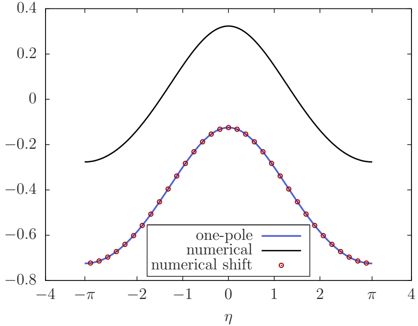
$$\frac{\partial F}{\partial \tau} = U^- - \frac{1}{2}\left(\widetilde{\nabla} F\right)^2 + \delta \mathrm{M_a} \widetilde{\nabla}^2 F$$

- $\mathcal{B}_a(\tau) = \left[\!\left[\dfrac{\partial p_a}{\partial \widetilde{\xi}} \right]\!\right]_-^+ = \left[\!\left[-R\dfrac{\partial u_a}{\partial \tau} \right]\!\right]_-^+$

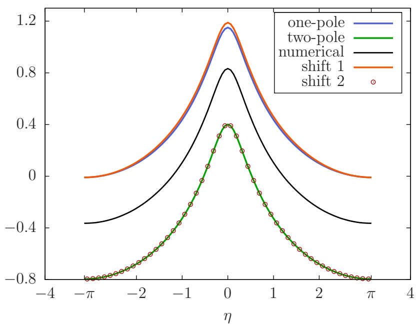
- Acoustic-flow coupling
- Flow-flame coupling

3 Steady state solutions and linear instability

- Steady solution $\leftrightarrow$ solution of steady Michelson-Sivashinsky
- Free parameter $\gamma = \dfrac{q}{\delta \mathrm{M_a}}$
- Results agree with analytical results of the theory of N-pole solutions.
  [Vaynblat & Matalon, SIAM J. Appl. Math., 2000]

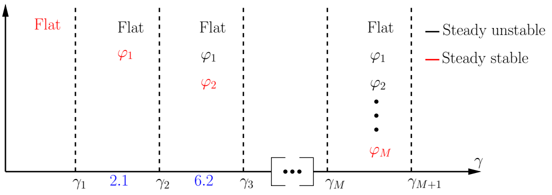


$\gamma = 2.1$

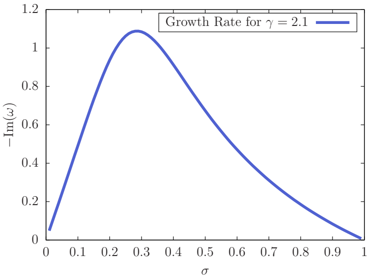


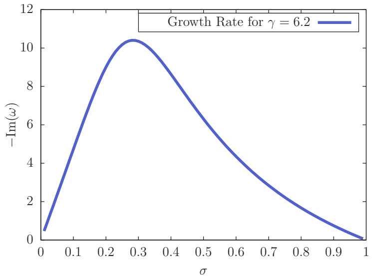
$\gamma = 6.2$

- No acoustic considerations [Vaynblat & Matalon, SIAM J. Apl. Math., 2000]



- Considering acoustics : linear instability!





4 **The one-equation model**

Using Fourier transforms $\widehat{\ }$ in the η and ζ direction, we can solve the hydrodynamic system analytically. We can then reduce the hydro equations, hydro jumps and flame equation to one equation in the spectral space

$$A\frac{\partial^2 \widehat{F}}{\partial \tau^2} + B(\boldsymbol{k})\frac{\partial \widehat{F}}{\partial \tau} + C(\boldsymbol{k}, \mathcal{B}_a(\tau))\widehat{F} = -|\boldsymbol{k}|\left(\widehat{\tilde{\nabla} F} \star \widehat{\tilde{\nabla} F}\right)(\boldsymbol{k}) - A\left(\widehat{\tilde{\nabla} F} \star \frac{\partial \widehat{\tilde{\nabla} F}}{\partial \tau}\right)(\boldsymbol{k}),$$

where A, B and C are functions known explicitly and $\star$ represents the convolution.

The whole problem reduces to two subproblems:

1. The acoustic system
2. The spectral flame equation

5 2D unsteady numerical approach

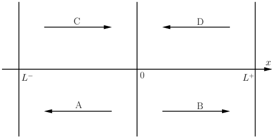
System of PDEs with discontinuous coefficient

$$\begin{cases} (p_a)_\tau + (u_a)_{\tilde{\xi}} & = & 0 \\ (u_a)_\tau + c^2 (p_a)_{\tilde{\xi}} & = & 0 \end{cases}, \quad \text{where} \quad c = \begin{cases} c^- & \text{if} & \tilde{\xi} < 0 \\ c^+ & \text{if} & \tilde{\xi} > 0 \end{cases}$$

Boundary and jump conditions

$$\begin{cases} \tilde{\xi} \in [L^-, L^+] \\ \tau \geqslant 0 \end{cases}, \quad \begin{cases} u_a(L^-, \tau) = 0 \\ p_a(L^+, \tau) = 0 \end{cases}, \quad \begin{cases} [\![u_a]\!]_-^+ = \mathcal{J}_a(\tau) \\ [\![p_a]\!]_-^+ = 0 \end{cases}$$

- Solved by semi-analytical method of characteristics



- So if we know $\mathcal{J}_a(\tau)$, we can solve the acoustic problem...

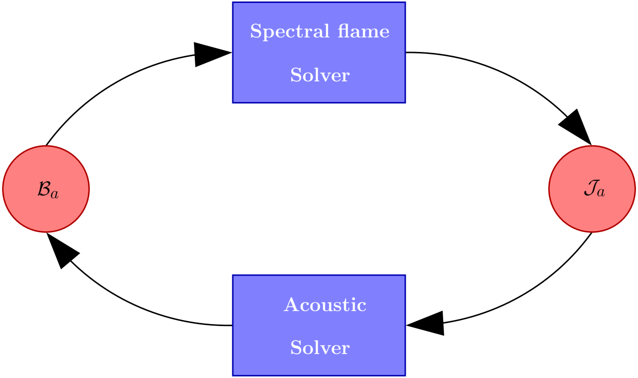
The spectral flame equation

$$
\begin{aligned}
A\frac{\partial^2 \widehat{F}}{\partial \tau^2} + B(\boldsymbol{k})\frac{\partial \widehat{F}}{\partial \tau} + C(\boldsymbol{k}, \mathcal{B}_a(\tau))\widehat{F} \quad &= \quad -|\boldsymbol{k}|\left(\widehat{\widetilde{\nabla} F} \star \widehat{\widetilde{\nabla} F}\right)(\boldsymbol{k}) \\
&- \quad A\left(\widehat{\widetilde{\nabla} F} \star \frac{\partial \widehat{\widetilde{\nabla} F}}{\partial \tau}\right)(\boldsymbol{k}),
\end{aligned}
$$

$$
\widehat{F}(k,0) = \widehat{F_0}(k) \quad \text{and} \quad \frac{\partial \widehat{F}}{\partial \tau}(k,0) = 0
$$

- Use FFT and IFFT to evaluate the convolutions
- March in time using e.g. 4th-order Adams-Bashforth
- So in theory if we know $\mathcal{B}_a(\tau)$, the flame equation can be solved...

- The whole system can be solved by coupling the two methods



6 Numerical results

- Constants of the problem

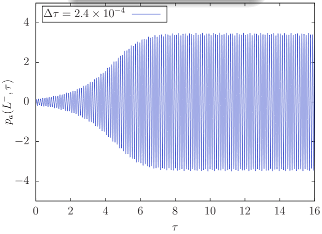
Parameters	σ	M	U_L	q	ℓ^*	h^*	G
Values	0.5	0.0007	0.24 m $\cdot$ s^{-1}	5.25	1.2 m	0.1 m	0

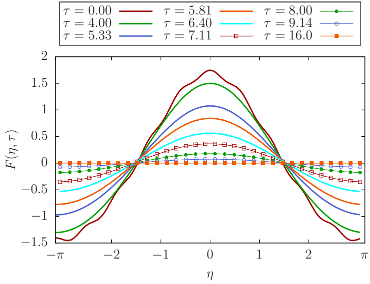
- Results presented for two values of the "free" parameter γ:

$\gamma = 2.1$ $\qquad$ $\gamma = 6.2$

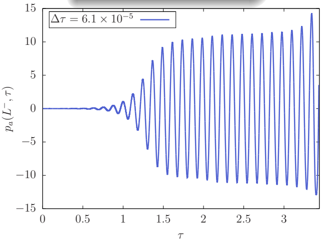
Acoustic pressure and flame shape

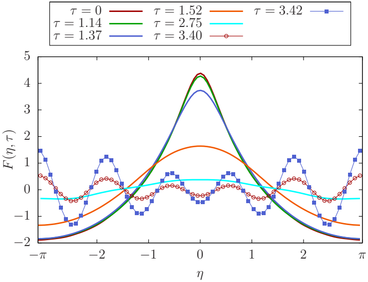
$\gamma = 2.1$





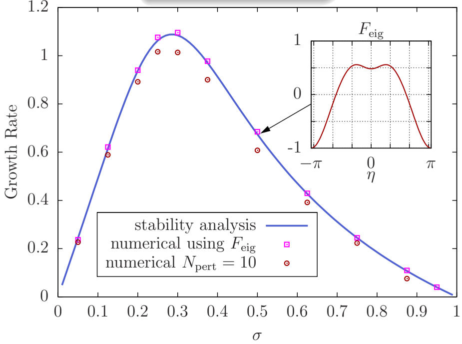
$\gamma = 6.2$



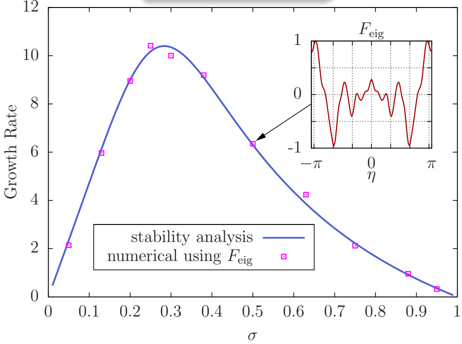


Validation 1: growth rate

$\gamma = 2.1$

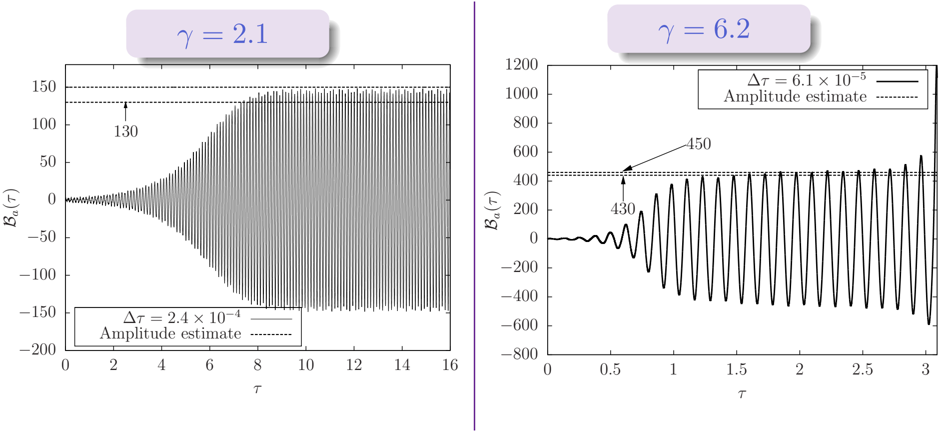


$\gamma = 6.2$



Agreement between numerics and linear stability analysis

Validation 2: nonlinear behaviour 1/2

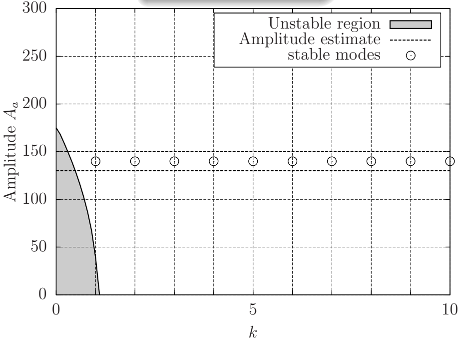


- ω_1 dominant + pressure saturation $\rightarrow \mathcal{B}_a(\tau) \approx A_a \cos(\omega_1 \tau)$
- flattening of the flame $\rightarrow$ linearisation of the flame equation around flat state

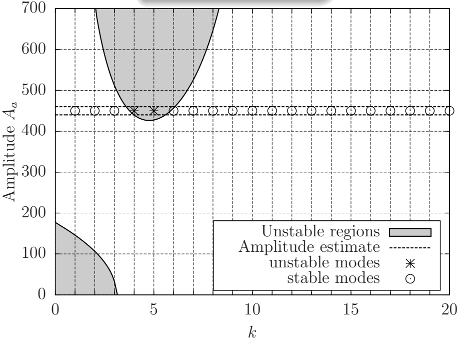
Simplified spectral flame equation: damped Mathieu Equation

$$\frac{\partial^2 \hat{F}}{\partial \tau^2} + \nu^*(k)\frac{\partial \hat{F}}{\partial \tau} + \left[\delta^*(k) + \epsilon^*(k)\cos(\omega_1 \tau)\right]\hat{F} \quad = \quad 0$$

$\gamma = 2.1$



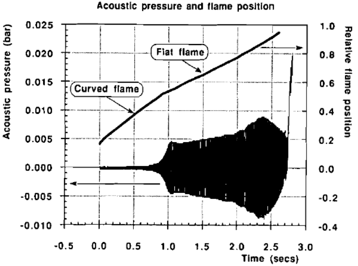
$\gamma = 6.2$

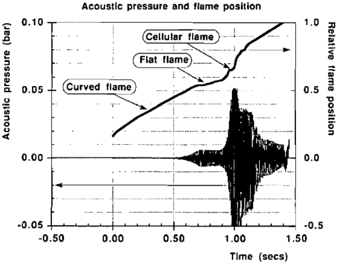


- The curved steady states are linearly unstable.
- For reasonably low values of γ, a flat flame (intrinsically unstable in silent environment) can survive in a noisy (spontaneous) environment.
- For larger values of γ, a cellular flame is forming, corresponding to a weakly-nonlinear instability (subharmonic parametric).
- Both cases correspond qualitatively to experimental observations

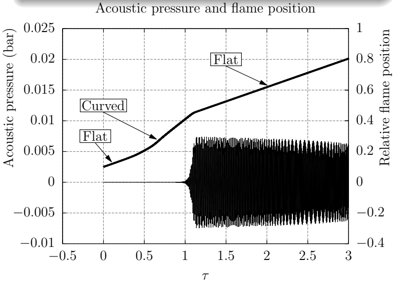
Propagating flame with gravity

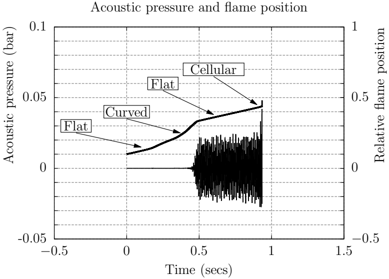
Experiment [Searby, 1992]





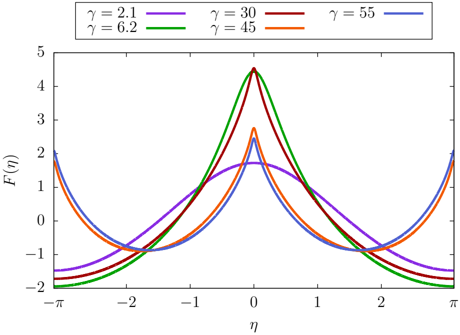
Numerics



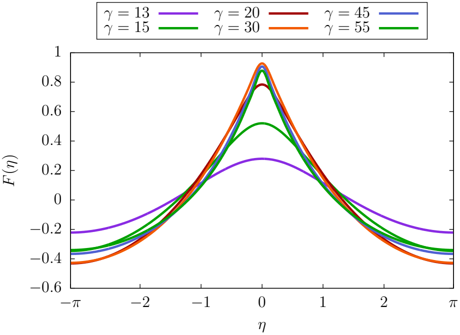


Towards better agreement? Hint from steady states

Current



Full model



For similar values of γ, the steady states of the full model are less cusped and more compact.

7 Instability triggering by vortical disturbances

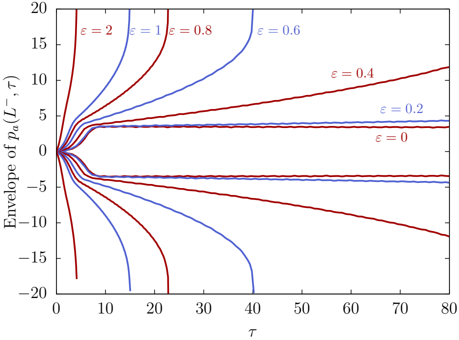
- Periodic forcing of hydrodynamic velocity U

Spectral flame equation

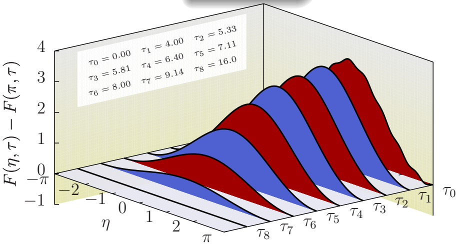
$$\begin{aligned} A\frac{\partial^2 \hat{F}}{\partial \tau^2} + B(k)\frac{\partial \hat{F}}{\partial \tau} + C(k,\tau)\hat{F} &= -|k|\left(iu\hat{F}(u)\right) \star \left(iu\hat{F}(u)\right)(k) \\ &- A\left(iu\hat{F}(u)\right) \star \left(iu\frac{\partial \hat{F}}{\partial \tau}(u)\right)(k) \\ &+ \mathcal{N}_0(\omega, k, k_0, \tau, \varepsilon) \end{aligned}$$

- ω: frequency of the disturbance
- k_0: wavenumber of the disturbance
- ε: amplitude of the disturbance

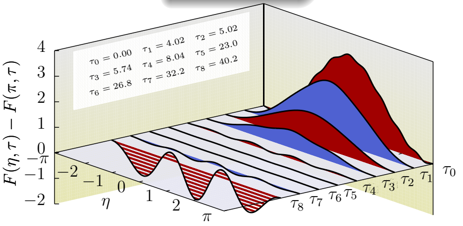
$\gamma = 2.1$ (was "stable" wihtout vortical disturbances)



$\varepsilon = 0$

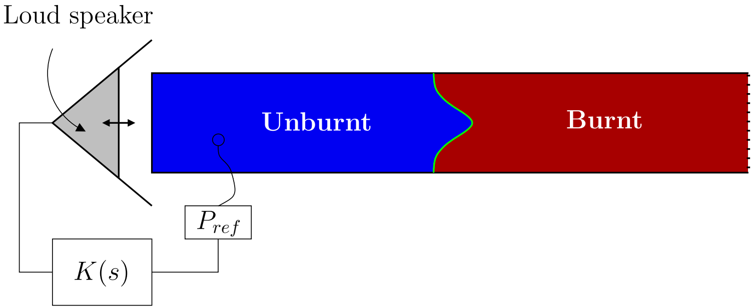


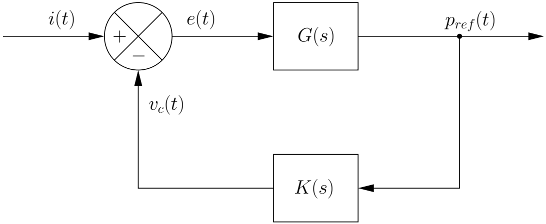
$\varepsilon = 0.6$



8 Feedback control of combustion instabilities

Feedback control implementation





The controller: 1st order phase compensator

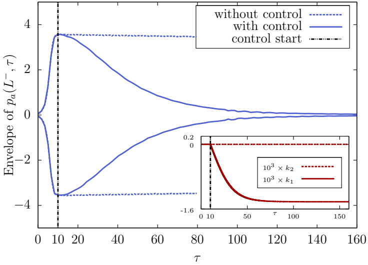
$$
\begin{aligned}
K(s) &= K(s,t) \\
&= k_1(t)\,\frac{s+z_c}{s+z_c+k_2(t)}
\end{aligned}
$$

Update rules

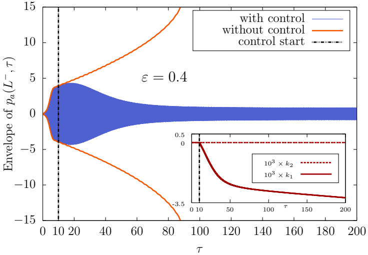
$$
\begin{aligned}
k_1(t) &= -\gamma_1 \int_0^t \left(p_{\mathrm{ref}}(\tau)\right)^2 \mathrm{d}\tau \\
k_2(t) &= +\gamma_2 \int_0^t p_{\mathrm{ref}}(t')\, k_1(t')\, J(t')\, \mathrm{d}t' \\
J(t) &= \int_0^t p_{\mathrm{ref}}(\tau) \exp\left\{-\left[z_c + k_2(t-\tau)\right](t-\tau)\right\} \mathrm{d}\tau
\end{aligned}
$$

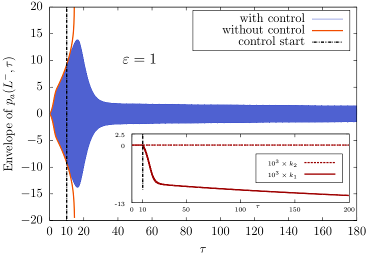
Reduction of instability effect 1/2

- Without vortical disturbances

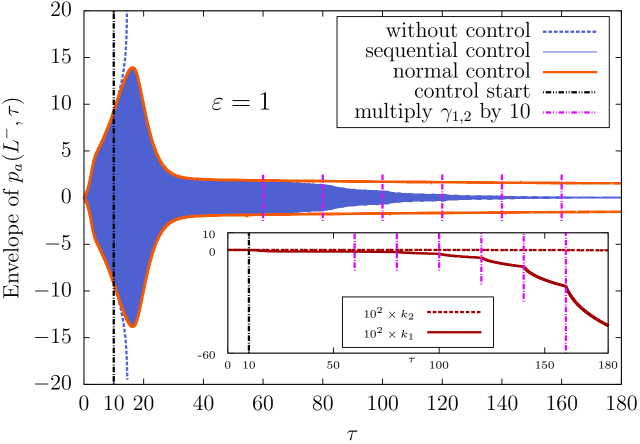


- With vortical disturbances





- New strategy to "kill" the signal faster: sequential control
- Changing the convergence rate when gradient is "calm"
- $\gamma_{1,2} \to 10 \times \gamma_{1,2}$



Summary [RCA & Wu, JFM, 2014]

- Implementation of "complete" flame model
- Analytical linear stability analysis of curved flames
- Unsteady coupled numerical scheme

Other things that have been done [RCA & Wu, AIAA, 2014]

- Modelling effect of weak turbulence in fresh mixture
- Use adaptive feedback control to suppress instabilities
- Implementation of $O(\delta)$ model

Future work and challenges

- Refining model to get *quantitative* agreement with experiments
- Analyse the effect of not making a weak nonlinear assumption
- Implementation of acoustic loss in the system