Introduction to thermoacoustics theory

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1. Introduction

1.1. Basic assumptions and conservation equations

Consider a fluid with

v : velocity, has components v_i $i = 1, 2, 3$

ρ : density

p : pressure

σ : viscous stress tensor, has components σ_{ij} , $i, j = 1, 2, 3$

where $$\sigma_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}\right)$$

η : coefficient of shear viscosity

conservation equation for mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0$$

conservation equation for momentum

(body forces such as gravity ignored)

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nabla \cdot \sigma$$

or

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}$$

1.2. Thermodynamic relationships

Consider a fluid element with constant mass m, undergoing a reversible process (heat added, volume changed).

change of internal energy: $$\mathrm{d}U = \underbrace{\mathrm{d}H}_{\text{heat energy added}} - \underbrace{p\,\mathrm{d}V}_{\text{work done on fluid element}}$$

change of entropy: $$\mathrm{d}S = \frac{\mathrm{d}H}{T}$$

Introduce the specific quantities, i.e. quantities per unit mass:

internal energy	heat energy	entropy
$e = \dfrac{U}{m}$	$h = \dfrac{H}{m}$	$s = \dfrac{S}{m}$

$\Rightarrow$ specific entropy: $\mathrm{d}s = \dfrac{1}{T}\mathrm{d}h$

mass density (definition): $\rho = \dfrac{m}{V}$ $\Rightarrow$ $\dfrac{1}{\rho} = \dfrac{V}{m}$

$\Rightarrow$ specific internal energy: $\mathrm{d}e = \mathrm{d}h - p\,\mathrm{d}(\dfrac{1}{\rho}) = T\,\mathrm{d}s - p\,\mathrm{d}(\dfrac{1}{\rho})$

specific enthalpy (definition): $B = e + \dfrac{p}{\rho}$

$\Rightarrow$ change of the specific enthalpy of the fluid element:

$$\mathrm{d}B = \mathrm{d}e + \mathrm{d}(\frac{p}{\rho}) = \mathrm{d}e + p\,\mathrm{d}(\frac{1}{\rho}) + \frac{1}{\rho}\mathrm{d}p = T\mathrm{d}s + \frac{1}{\rho}\mathrm{d}p$$

specific heat capacities (ratio of added heat to corresponding increase in temperature):

$$c_V = \left(\frac{\mathrm{d}h}{\mathrm{d}T}\right)_\rho = \left(\frac{\partial e}{\partial T}\right)_\rho = T\left(\frac{\partial s}{\partial T}\right)_\rho, \quad c_p = \left(\frac{\mathrm{d}h}{\mathrm{d}T}\right)_p = \left(\frac{\partial B}{\partial T}\right)_p = T\left(\frac{\partial s}{\partial T}\right)_p$$

When heat is added to the fluid element, the density of the fluid becomes a function of two variables, the pressure and specific entropy, say: $\rho=\rho(p,s)$. Hence

$$d\rho = \underbrace{\left(\frac{\partial\rho}{\partial p}\right)_s}_{=\frac{1}{c^2}} \mathrm{d}p + \underbrace{\left(\frac{\partial\rho}{\partial s}\right)_p}_{=-\frac{\rho}{c_p}} \mathrm{d}s$$

for an ideal gas

because: $\rho = \dfrac{p}{RT}$, so $\left(\dfrac{\partial\rho}{\partial s}\right)_p = \underbrace{\left(\dfrac{\partial\rho}{\partial T}\right)_p}_{=-\frac{p}{RT^2}} \underbrace{\left(\dfrac{\partial T}{\partial s}\right)_p}_{=\frac{T}{c_p}} = -\dfrac{\rho}{c_p}$

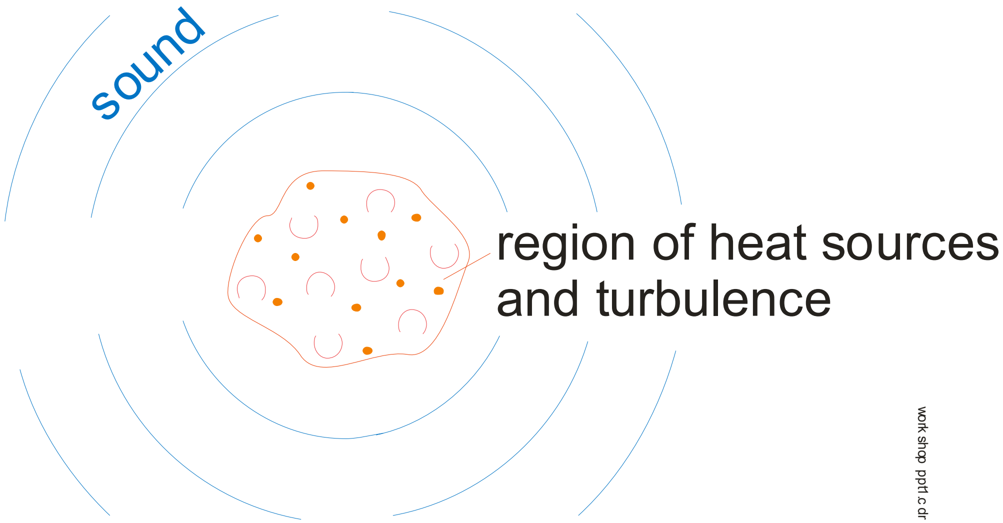
or

$$\boxed{\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c^2}\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\rho}{c_p}\frac{\mathrm{D}s}{\mathrm{D}t}}$$

2. Lighthill’s acoustic analogy

2.1. Derivation

Consider an unbounded fluid with a finite source region containing vortices and combustion.



fluid at rest

ρ_0, p_0

uniform
mean density
and
mean pressure

Aim: derive a governing equation for the pressure *p* from the two conservation equations for mass and momentum from mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0 \quad \Bigg| + \frac{1}{c_0^2}\frac{\partial p}{\partial t} \qquad \text{where} \qquad c_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

c_0 : will turn out to be the speed of sound

subscript s : isentropic process, i.e. no heat added

$$\Rightarrow \qquad \frac{1}{c_0^2}\frac{\partial p}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = \frac{\partial}{\partial t}\left(\frac{p}{c_0^2} - \rho\right)$$

from momentum conservation:

$$\rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} \qquad \Bigg| + v_i \frac{\partial \rho}{\partial t} + v_i \frac{\partial \rho v_j}{\partial x_j} = 0$$

$$\Rightarrow \qquad \frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} \qquad \text{Reynold's form}$$

mass $$\frac{1}{c_0^2}\frac{\partial p}{\partial t}+\frac{\partial \rho v_j}{\partial x_j}=\frac{\partial}{\partial t}\left(\frac{p}{c_0^2}-\rho\right) \quad \Big| \frac{\partial}{\partial t}$$

momentum $$\frac{\partial \rho v_i}{\partial t}+\frac{\partial \rho v_i v_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial \sigma_{ij}}{\partial x_j} \quad \Big| \frac{\partial}{\partial x_i}$$

differentiate as indicated and subtract equations to eliminate

the momentum density ρv_i

$$\Rightarrow \quad \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\frac{\partial^2 p}{\partial x_i \partial x_i}=\frac{\partial^2}{\partial t^2}\left(\frac{p-p_0}{c_0^2}-(\rho-\rho_0)\right)+\frac{\partial^2(\rho v_i v_j-\sigma_{ij})}{\partial x_i \partial x_j}$$

p_0, ρ_0 inserted constants (convention)

$$\rho_e=\frac{p-p_0}{c_0^2}-(\rho-\rho_0)$$ excess density

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\frac{\partial^2 p}{\partial x_i \partial x_i}=\frac{\partial^2}{\partial t^2}\left(\frac{p-p_0}{c_0^2}-(\rho-\rho_0)\right)+\frac{\partial^2(\rho v_i v_j-\sigma_{ij})}{\partial x_i \partial x_j}$$

This is an exact equation: no approximations (e.g. linearisation) have been made.

If

- there is no source region,
- the motion is linear,
- viscosity effects are neglected,

then it reduces to

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2}-\frac{\partial^2 p'}{\partial x_i \partial x_i}=0.$$

Wave equation for the acoustic pressure p'

c_0: the propagation speed of acoustic waves.

A governing equation for the density ρ can be obtained in the same way.

Just add $c_0^2 \frac{\partial p}{\partial t}$ to either side of the momentum equation, instead of adding $\frac{1}{c_0^2}\frac{\partial p}{\partial t}$ to the mass equation.

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i \partial x_i} = \frac{\partial^2 [\rho v_i v_j - \sigma_{ij} + ((p - p_0) - c_0^2(\rho - \rho_0)\delta_{ij}]}{\partial x_i \partial x_j}$$

This is **Lighthill's acoustic analogy equation**.

The term $\rho v_i v_j - \sigma_{ij} + ((p - p_0) - c_0^2(\rho - \rho_0)\delta_{ij}$
on the right hand side is called **Lighthill stress tensor**.

2.2. Interpretation

Analogy equation:

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}-\frac{\partial^2 p}{\partial x_i \partial x_i}=\frac{\partial^2 \rho_e}{\partial t^2}+\frac{\partial^2(\rho v_i v_j-\sigma_{ij})}{\partial x_i \partial x_j}$$

With $$\frac{D\rho}{Dt}=\frac{1}{c^2}\frac{Dp}{Dt}-\frac{\rho}{c_p}\frac{Ds}{Dt}$$ (see section 1.2)

and a few straightforward mathematical manipulations, the source term $\frac{\partial \rho_e}{\partial t}$ can be expressed in terms of the entropy:

$$\frac{\partial\rho_e}{\partial t} = -\frac{\rho_0}{c_p}\frac{Ds}{Dt} - \frac{\partial\rho_e u_i}{\partial x_i} - \frac{1}{c_0^2}\left(1-\frac{\rho_0 c_0^2}{\rho c^2}\right)\frac{Dp}{Dt} + \frac{p-p_0}{c_0^2}\frac{D\rho}{Dt}$$

and the analogy equation then becomes

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p =$$

$$\underbrace{\frac{\partial}{\partial t}\frac{\rho_0}{c_p}\frac{Ds}{Dt}}_{\text{entropic term}} + \underbrace{\frac{1}{c_0^2}\left(1-\frac{\rho_0 c_0^2}{\rho c^2}\right)\frac{Dp}{Dt} - \frac{p-p_0}{c_0^2}\frac{D\rho}{Dt} - \frac{\partial\rho_e u_i}{\partial x_i}}_{\text{indirect combustion noise}} + \underbrace{\frac{\partial^2(\rho v_i v_j - \sigma_{ij})}{\partial x_i \partial x_j}}_{\text{Lighthill's quadrupole term}}$$

$$\frac{\partial}{\partial t}\frac{\rho_0}{c_p}\frac{Ds}{Dt}+\frac{1}{c_0^2}(1-\frac{\rho_0 c_0^2}{\rho c^2})\frac{Dp}{Dt}-\frac{p-p_0}{c_0^2}\frac{D\rho}{Dt}-\frac{\partial \rho_e u_i}{\partial x_i}+\frac{\partial^2(\rho v_i v_j-\sigma_{ij})}{\partial x_i \partial x_j}$$

The analogy equation describes the effect of all the thermo-acoustic sources:

$$\frac{\partial}{\partial t}\frac{\rho_0}{c_p}\frac{Ds}{Dt}$$

entropic term related to heat release rate: $\frac{\rho}{c_p}\frac{Ds}{Dt}=(\frac{\rho}{c_p}\frac{q}{T})$, where $q=\frac{Dh}{Dt}$ is the rate of heat energy added to the fluid per unit mass

$$\frac{1}{c_0^2}(1-\frac{\rho_0 c_0^2}{\rho c^2})\frac{Dp}{Dt}-\frac{p-p_0}{c_0^2}\frac{D\rho}{Dt}$$

significant if there are regions of unsteady flow with different mean density and sound speed

$$\frac{\partial \rho_e u_i}{\partial x_i}$$

describes effect of momentum changes of density inhomogeneities

3. Rayleigh's criterion

3.1. Balance equation for the acoustic energy

A balance equation for the **acoustic energy** can be derived from the conservation equations for mass and momentum.

Assumptions:

$$p = p_0 + p', \quad p_0 = \text{const}$$

$$\rho = \rho_0 + \rho', \quad \rho_0 = \text{const}$$

$$v_i = u_i \quad \text{(no mean flow or vorticity)}$$

u_i : acoustic velocity

$$\sigma_{ij} = 0$$

mass $$\frac{1}{c_0^2}\frac{\partial p}{\partial t}+\frac{\partial \rho u_j}{\partial x_j}=\frac{\partial}{\partial t}\left(\frac{p}{c_0^2}-\rho\right) \quad \Bigg| \cdot \frac{p'}{\rho_0}$$

$$\frac{1}{\rho_0 c^2}\underbrace{p'\frac{\partial p'}{\partial t}}_{=\frac{1}{2}\frac{\partial p'^2}{\partial t}} + \frac{p'}{\rho_0}\underbrace{\frac{\partial \rho u_i}{\partial x_i}}_{\approx p'\frac{\partial u_i}{\partial x_i}} = \underbrace{\frac{p'}{\rho_0}\frac{\partial}{\partial t}\left(\frac{p}{c^2}-\rho\right)}_{=q'\frac{\rho_0}{c_p T}}$$

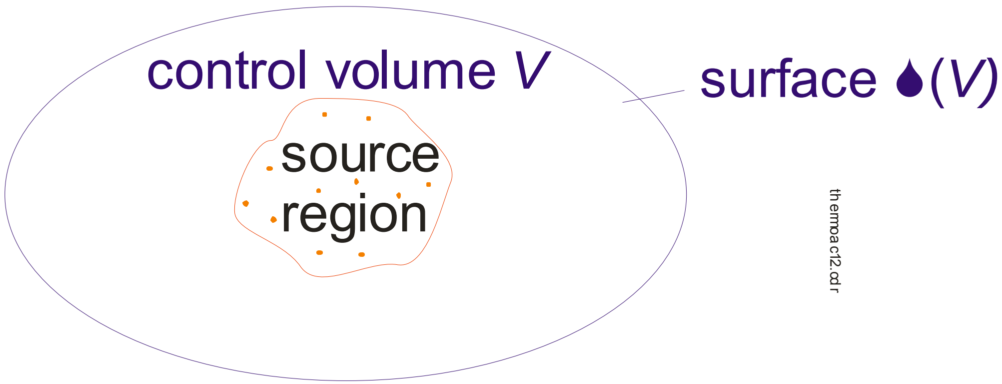
momentum $$\rho\frac{\partial u_i}{\partial t}+\rho u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i} \quad \Big| \cdot u_i$$

$$\underbrace{\rho u_i\frac{\partial u_i}{\partial t}}_{=\frac{1}{2}\frac{\partial u^2}{\partial t}}+\rho u_j \underbrace{u_i\frac{\partial u_i}{\partial x_j}}_{=\frac{1}{2}\frac{\partial u^2}{\partial x_j}}=-u_i\frac{\partial p}{\partial x_i}$$

add the two equations:

$$\frac{\partial}{\partial t}\left(\underbrace{\frac{1}{2}\frac{1}{\rho_0 c^2}p'^2}_{\substack{\text{potential}\\ \text{energy}\\ \text{density}\\ e_p}} + \underbrace{\frac{1}{2}\rho_0 u^2}_{\substack{\text{kinetic}\\ \text{energy}\\ \text{density}\\ e_k}}\right) + \frac{\partial}{\partial x_i}\underbrace{\left(p'u_i\right)}_{\substack{\text{acoustic}\\ \text{energy}\\ \text{flow}}} = \underbrace{\frac{p'q'}{c_p T}}_{\substack{\text{rate of energy}\\ \text{gain per unit}\\ \text{volume}}}$$

This is the local form of the conservation equation for the acoustic energy. To get the global form, integrate it over a fixed control volume V, which contains the heat sources.



$$\frac{\partial}{\partial t}\left(\int_V e_p \, \mathrm{d}V + \int_V e_k \, \mathrm{d}V\right) = \oiint_{S(V)} p' u_i \, \mathrm{d}S_i + \frac{1}{c_p T}\int_V p' q \, \mathrm{d}V$$

rate of change of acoustic energy E

loss of acoustic energy at the surface

gain of acoustic energy from the heat source

The system is unstable if E grows, averaged over a long time.

If losses are absent, instability occurs when $\int_V \langle p'q \rangle \, \mathrm{d}V > 0$.

time average

The balance equation can be extended to:

- case with mean flow (Chu's disturbance energy)
- rotational flow

3.2. Rayleigh's criterion

$$\text{instability if} \quad \int_V \underbrace{\langle p'q \rangle}_{\text{time average}} \, \mathrm{d}V > 0 \, .$$

The time average $\langle p'q \rangle$ is positive if the pressure and rate of heat release are in phase.

Note

Rayleigh’s criterion is a simplified form of the acoustic energy balance; it neglects energy losses, such as radiation from openings, losses in boundary layers, and vorticity.

Thank you!

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