Introduction to Green's functions

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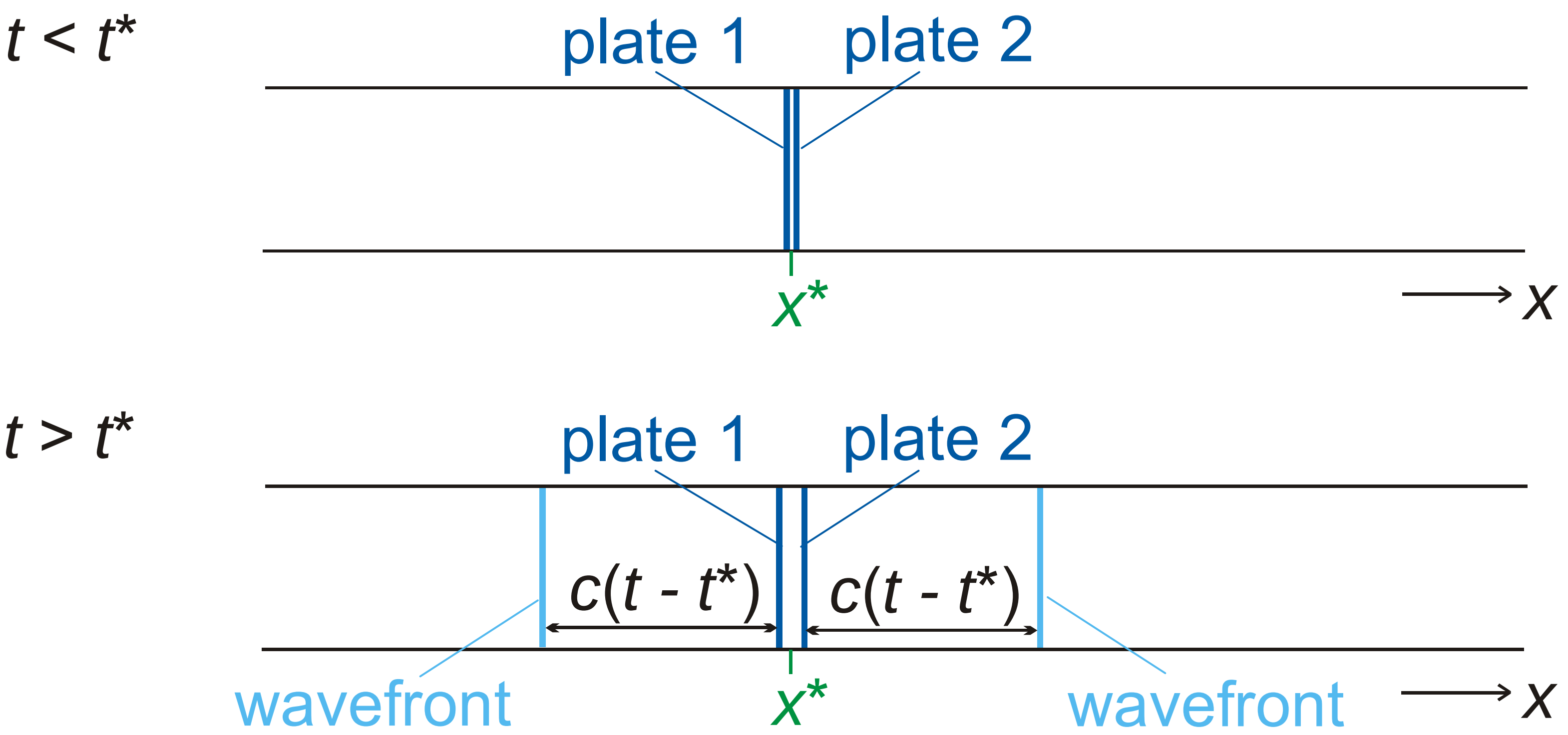
1. Free-space Green's function
2. Tailored Green's function
3. Green's function in the frequency domain
4. Compact Green's function

1. Free-space Green's function

1.1. General concept (in 1-D)

Consider: infinitely long tube
filled by compressible fluid with speed of sound c
two thin plates at x^*, moving apart abruptly at time t^*



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Green's function: response of the fluid to impulsive point source

depends on x^*: source position

t^*: firing time

x: observer position

t: observer time

notation: $G(x,t;\, x^*,t^*)$

The Green's function can be given in terms of various physical quantities, e.g.

sound pressure p'

acoustic velocity u'

velocity potential Φ

Properties of the Green's function

causality: no response before the impulse, i.e.

$$G(x,t;\, x^*,t^*) = 0 \quad \text{for} \quad t < t^*$$

$$\rightarrow G(x,t;\, x^*,t^*) = G(x,x^*,t - t^*)$$

reciprocity: same signal if source and receiver are swapped over, i.e.

$$G(x,x^*,t-t^*) = G(x^*,x,t-t^*)$$

governing equation:

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} = \delta(x - x^*)\,\delta(t - t^*)$$

forcing at point x^*

forcing at time t^*

Solution of the governing equation
(from theory of generalised functions)

backward travelling wave

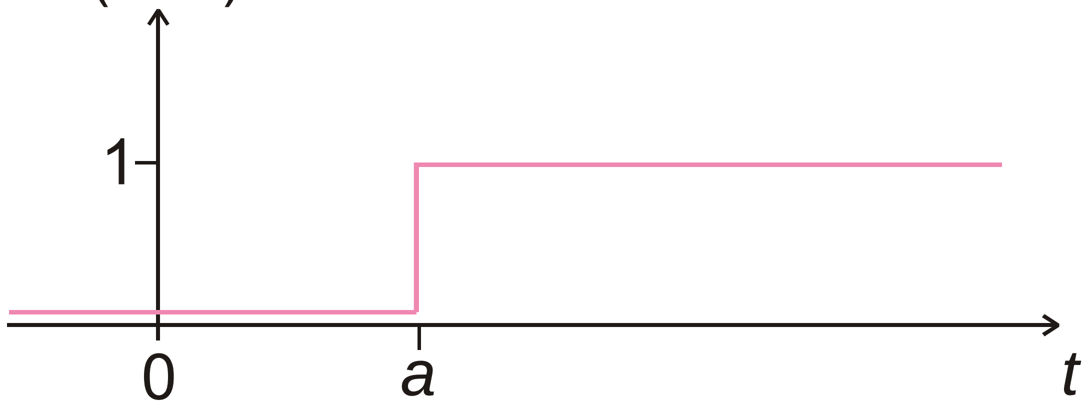
$$G(x,x^*,t-t^*) = \begin{cases} \dfrac{c}{2}\mathrm{H}\left(t-t^*+\dfrac{x-x^*}{c}\right) & \text{for} \quad x < x^* \\[2ex] \dfrac{c}{2}\mathrm{H}\left(t-t^*-\dfrac{x-x^*}{c}\right) & \text{for} \quad x > x^* \end{cases} =$$

forward travelling wave

$$= \frac{c}{2}\mathrm{H}\left(t-t^*-\frac{|x-x^*|}{c}\right)$$

Heaviside function

$\mathrm{H}(t - a)$



Application of the Green's function

Building block for generating solutions of the acoustic wave equation with a source term $S(x,t)$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=S(x,t) \qquad \Big| \cdot G(x,x^*,t-t^*)$$

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}-\frac{\partial^2 G}{\partial x^2}=\delta(x-x^*)\delta(t-t^*) \qquad \Big| \cdot \phi(x^*,t^*)$$

Exploit reciprocity: $x \leftrightarrow x^*$

Multiply as indicated

Subtract resulting equations

Integrate on both sides: $\int\limits_{t^*=0}^{t} \int\limits_{x^*=-\infty}^{\infty} \ldots \, \mathrm{d}x^* \, \mathrm{d}t^*$

Result:

$$\phi(x,t) = \int_{t^*=0}^{t} \int_{x^*=-\infty}^{\infty} G(x,x^*,t-t^*)\, S(x^*,t^*)\, \mathrm{d}x^*\, \mathrm{d}t^*$$

This is the solution of the above PDE for Φ.

Special case: point source at x_S, i.e.

$$S(x,t) = S(t)\,\delta(x - x_S)$$

Then

$$\phi(x,t) = \int_{t^*=0}^{t} G(x,x_S,t-t^*)\, S(t^*)\, \mathrm{d}t^*$$

Summary

We know: Green's function $G(x, x^*, t - t^*)$

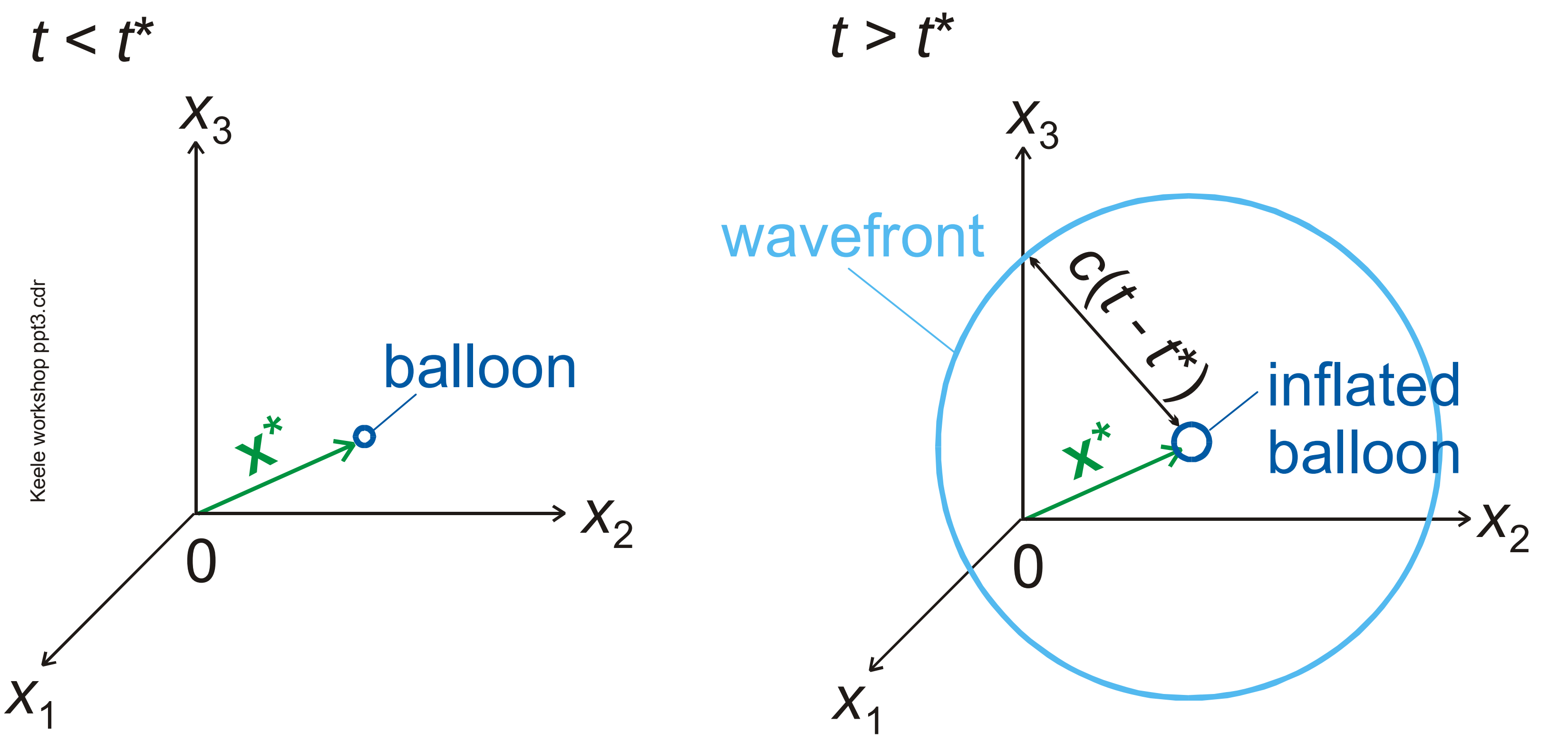
source distribution $S(x, t)$

We get: Solution of the PDE for Φ in terms of an integral over the source region and the forcing duration

$$\phi(x, t) = \int_{t^*=0}^{t} \int_{x^*=-\infty}^{\infty} G(x, x^*, t - t^*)\, S(x^*, t^*)\, \mathrm{d}x^*\, \mathrm{d}t^*$$

1.2. Extension to 3-D

Consider: infinitely extended 3-D region
filled by compressible fluid with speed of sound c
tiny balloon at **x***, inflating abruptly at time *t**



Green's function: $G(\mathbf{x},t;\, \mathbf{x}^*,t^*)$

depends on $\mathbf{x}^*$: source position

t^*: firing time

$\mathbf{x}$: observer position

t: observer time

Properties of the Green's function

causality: $G(\mathbf{x},\mathbf{x}^*,t-t^*) = 0$ for $t < t^*$

reciprocity: $G(\mathbf{x},\mathbf{x}^*,t-t^*) = G(\mathbf{x}^*,\mathbf{x},t-t^*)$

governing equation: $$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \delta(\mathbf{x}-\mathbf{x}^*)\delta(t-t^*)$$

Solution: $$G(\mathbf{x},\mathbf{x}^*,t-t^*) = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}^*|}\delta\left(t-t^*-\frac{|\mathbf{x}-\mathbf{x}^*|}{c}\right)$$

spherical wave travelling away from $\mathbf{x}^*$

Application of the Green's function

Building block for generating solutions of the acoustic wave equation with a source term $S(\mathbf{x},t)$

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi = S(\mathbf{x},t)$$

Solution

$$\phi(\mathbf{x},t) = \int_{t^*=0}^{t}\int_{V(\mathbf{x}^*)} G(\mathbf{x},\mathbf{x}^*,t-t^*)\, S(\mathbf{x}^*,t^*)\, \mathrm{d}^3\mathbf{x}^*\, \mathrm{d}t^*$$

volume enclosing the sources

Summary

We know: Green's function $G(\mathbf{x},\mathbf{x}^*,t-t^*)$

source distribution $S(\mathbf{x},t)$

We get: acoustic field generated by the source distribution

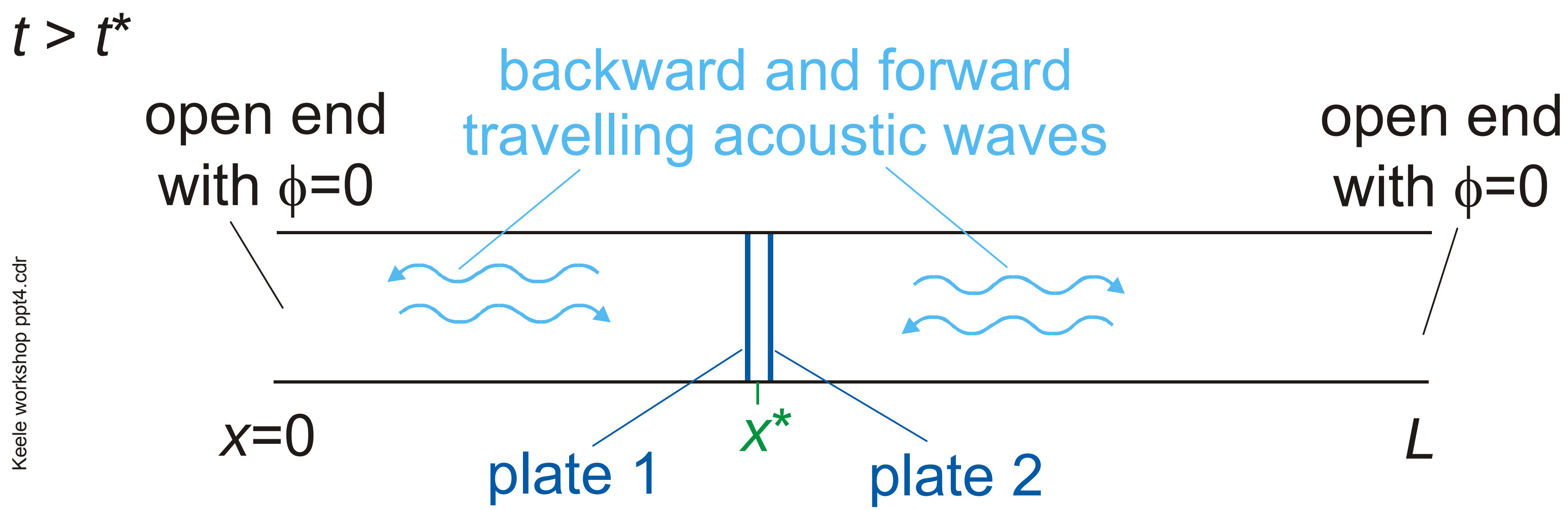
Note: This works in *free space* (1-D, 2-D and 3-D).

2. Tailored Green's function

also called "exact Green's function"

Idea: extend the building-block concept to fluids with boundaries

Example: 1-D tube with open ends



Tailored Green's function: response of the fluid to impulsive point source, with boundary conditions satisfied

Governing equations

PDE: $$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} = \delta(x - x^*)\delta(t - t^*)$$

bc's: $$G(x, x^*, t - t^*)\Big|_{x=0} = 0\,, \qquad G(x, x^*, t - t^*)\Big|_{x=L} = 0$$

(assuming that G is a velocity potential)

The **solution** is of the form

$$G(x, x^*, t - t^*) = \begin{cases} 0 & \text{for} \quad t < t^* \\ \sum_{n=1}^{\infty} G_n(x, x^*)\, e^{-i\omega_n (t - t^*)} & \text{for} \quad t > t^* \end{cases}$$

amplitudes frequencies

Superposition of modes

n: mode number

ω_n: eigenfrequency of mode n

G_n: Green's function amplitude of mode n

ω_n and G_n can be calculated from the PDE and bc's.

Results (calculation not shown)

$$\omega_n = \frac{n\pi c}{L}$$

$$G_n(x, x^*) = \begin{cases} \dfrac{(-1)^n}{n} \sin\dfrac{\omega_n x}{c} \sin\dfrac{\omega_n (x^* - L)}{c} & \text{for} \quad x < x^* \\ \dfrac{(-1)^n}{n} \sin\dfrac{\omega_n (x - L)}{c} \sin\dfrac{\omega_n x^*}{c} & \text{for} \quad x > x^* \end{cases}$$

Calculation possible for other simple cases, e.g.

- tube with cold and hot region
- tube with general end conditions
- tube with jump in cross-sectional area

Building block concept

The field in a tube with general source distribution $S(x,t)$ is described by:

PDE: $$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = S(x,t)$$

bc's: $$\phi(x,t)\Big|_{x=0} = 0\,, \qquad \phi(x,t)\Big|_{x=L} = 0$$

If $G(x,x^*,t - t^*)$ is known, the solution to these equations is

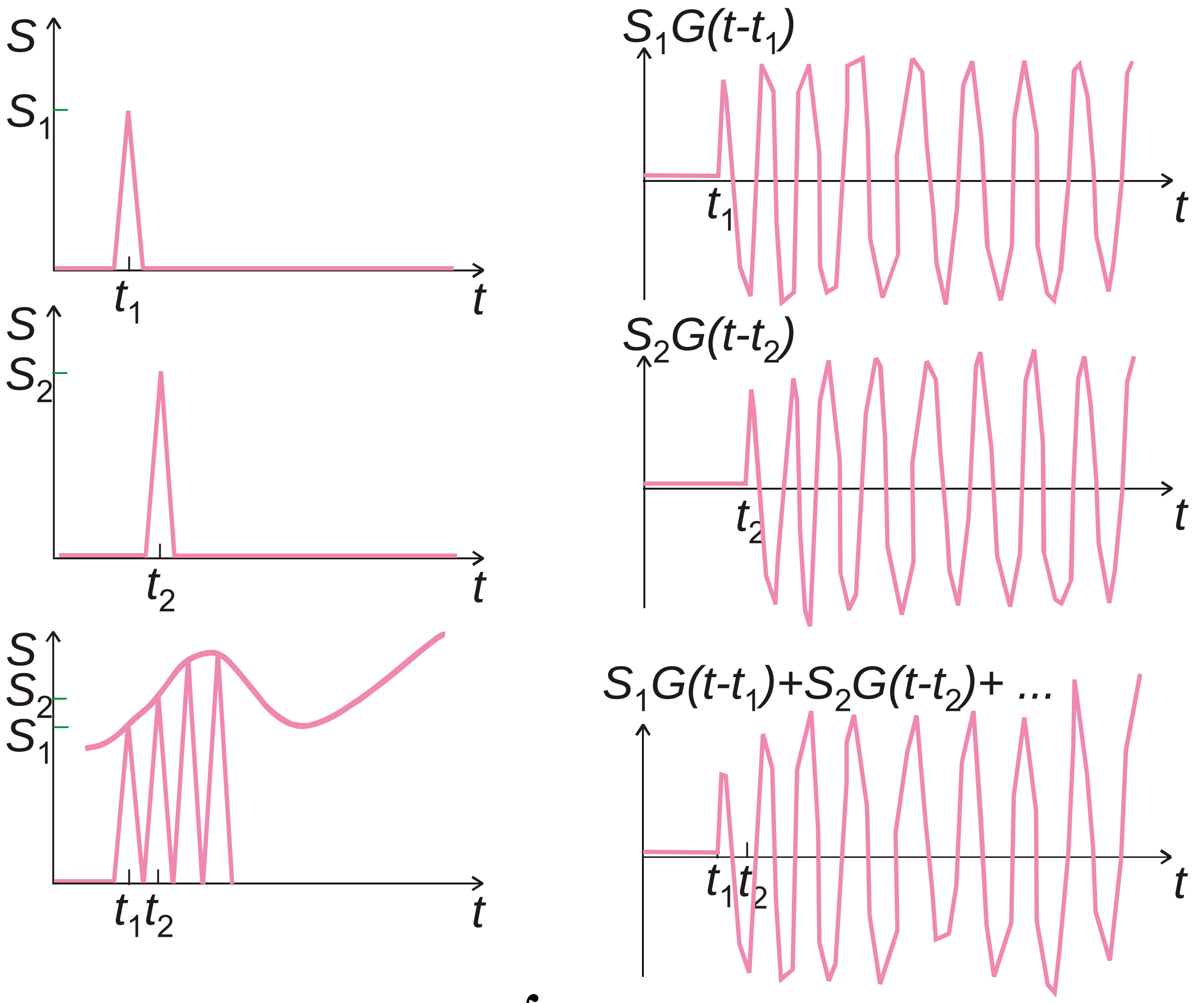
$$\phi(x,t) = \int_{t^*=0}^{t} \int_{x^*=-\infty}^{\infty} G(x,x^*,t-t^*)\, S(x^*,t^*)\, \mathrm{d}x^*\, \mathrm{d}t^*$$

Special case: Rijke tube, $S(x,t) = S(t)\,\delta(x - x_S)$

$$\phi(x,t) = \int_{t^*=0}^{t} G(x,x_S,t-t^*)\, S(t^*)\, \mathrm{d}t^*$$

position of the hot gauze

Physical interpretation of this equation



$$\sum_{n} S(t_n)\,G(t-t_n) \;\rightarrow\; \int_{t^*} S(t^*)\,G(t-t^*)\,dt^*$$

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Summary

The tailored Green’s function is the response of a fluid with boundaries (typically a fluid within a resonator).

It is “tailored” to the geometry of the resonator.

It is a superposition of resonator modes.

It is harder to calculate than the free-space Green’s function.

It can in principle be measured.

3. Green's function in the frequency domain

3.1. Free-space Green's function

1- D

Consider source distribution with harmonic time dependence (frequency ω)

$$S(x,t) = \hat{S}(x,\omega)\, e^{-i\omega t}$$

The resulting acoustic wave has the same time dependence:

$$\phi(x,t) = \hat{\phi}(x,\omega)\, e^{-i\omega t}$$

Then the governing equation for Φ,

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = S(x,t)$$

reduces to

$$(\frac{\omega}{c})^2\ \hat{\phi}(x,\omega) + \frac{\partial^2\hat{\phi}}{\partial x^2} = -\hat{S}(x,\omega)$$

If we put

$$G(x,x^*,t-t^*) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} \hat{G}(x,x^*,\omega)\, e^{-i\omega(t-t^*)}\, d\omega$$

then we can transform the governing equation for G,

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} = \delta(x-x^*)\,\delta(t-t^*)$$

from the time domain to the frequency domain:

$$\left(\frac{\omega}{c}\right)^2 \hat{G}(x,x^*,\omega) + \frac{\partial^2 \hat{G}}{\partial x^2} = -\delta(x-x^*)$$

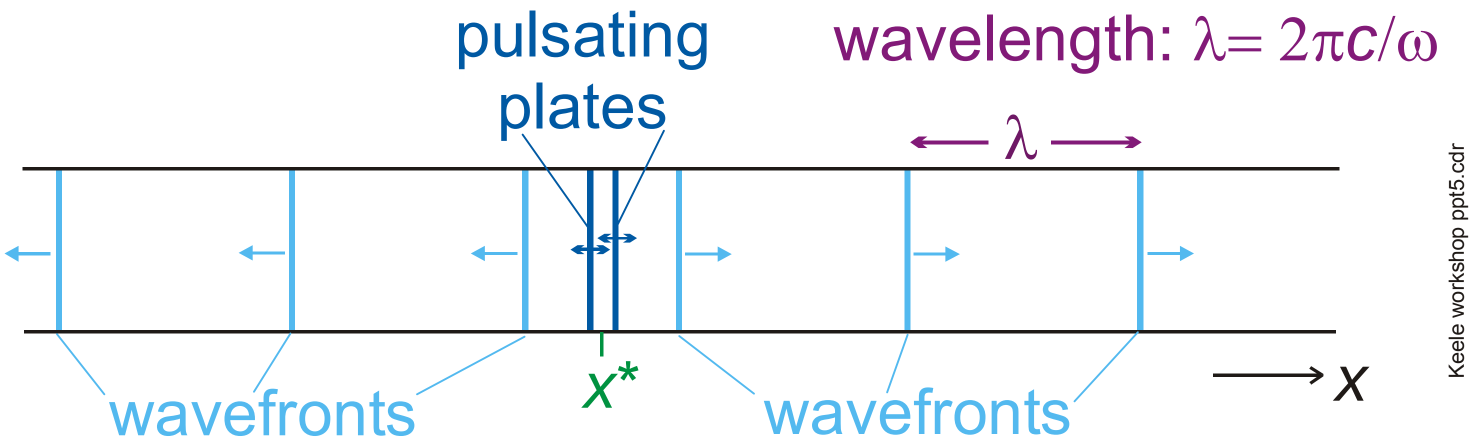
Solution:

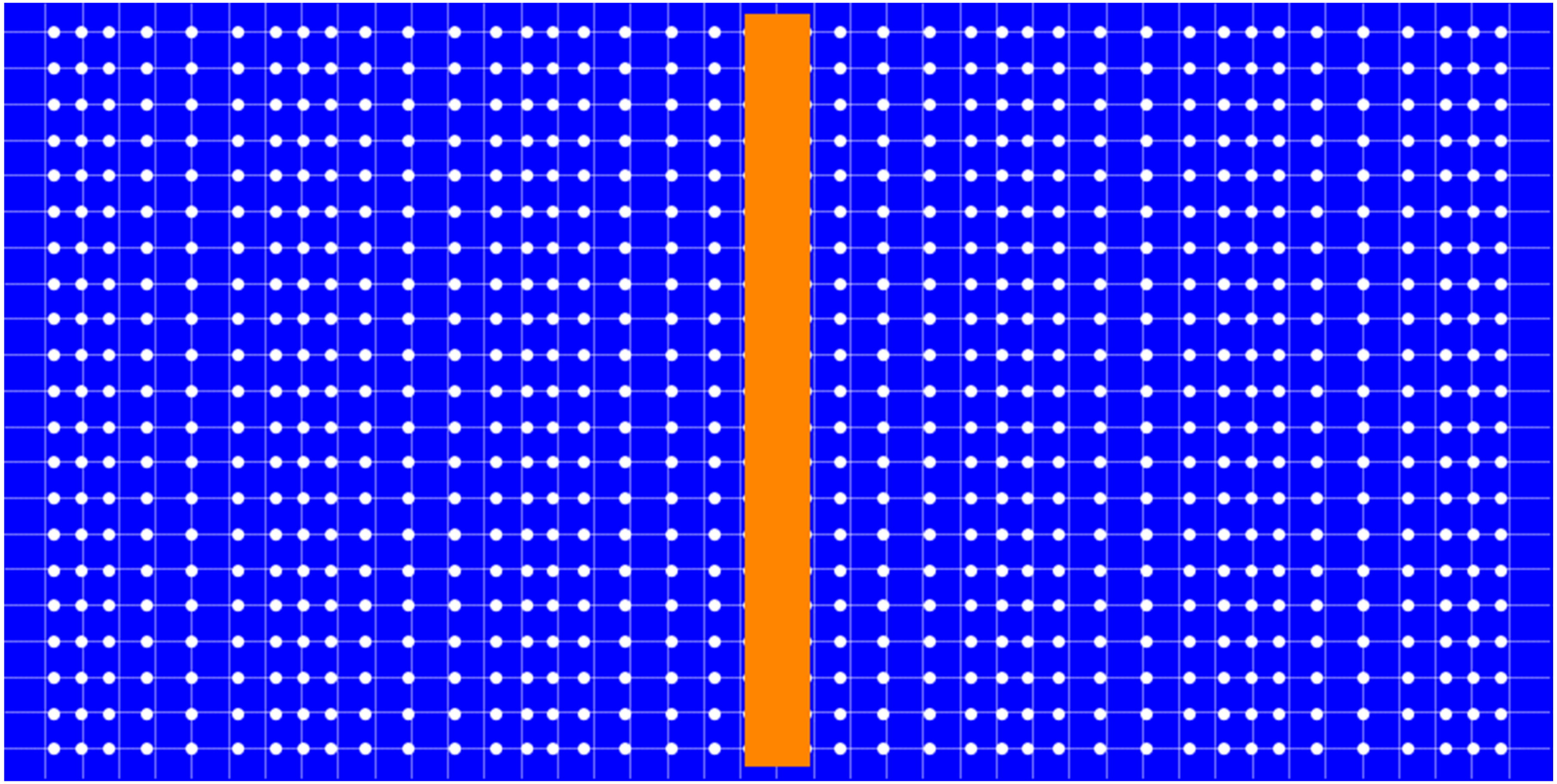
backward travelling wave

$$\hat{G}(x,x^*,\omega) = \begin{Bmatrix} \dfrac{i}{2\frac{\omega}{c}}\, e^{-i\frac{\omega}{c}(x-x^*)} & \text{for} \quad x < x^* \\ \dfrac{i}{2\frac{\omega}{c}}\, e^{+i\frac{\omega}{c}(x-x^*)} & \text{for} \quad x > x^* \end{Bmatrix} = \frac{i}{2\frac{\omega}{c}}\, e^{-i\frac{\omega}{c}|x-x^*|}$$

forward travelling wave

This is the acoustic wave generated by a pair of plates pulsating with frequency ω.





3- D

source distribution: $S(\mathbf{x},t) = \hat{S}(\mathbf{x},\omega)\, e^{-i\omega t}$

Governing equation for the velocity potential Φ in ω-domain:

$$\left(\frac{\omega}{c}\right)^2 \hat{\phi}(\mathbf{x},\omega) + \nabla^2 \hat{\phi} = -\hat{S}(\mathbf{x},\omega)$$

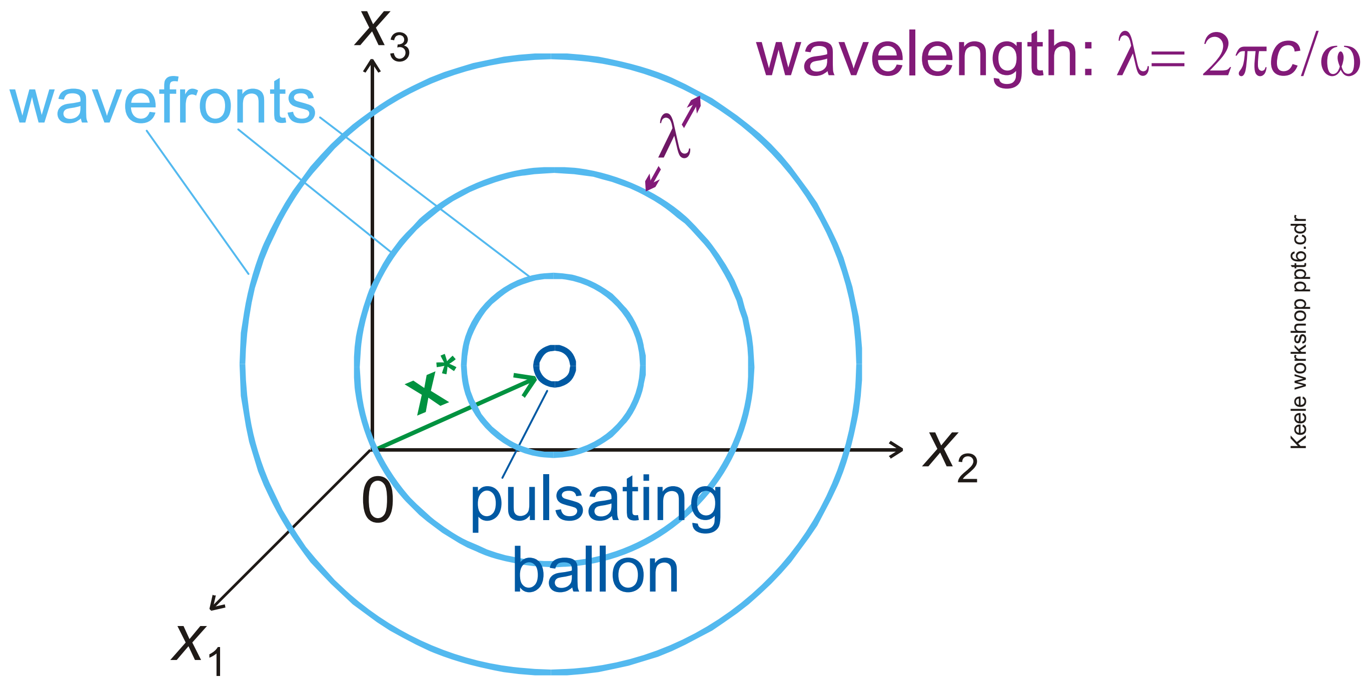
Governing equation for Green's function in ω-domain:

$$\left(\frac{\omega}{c}\right)^2 \hat{G}(\mathbf{x},\mathbf{x}',\omega) + \nabla^2 \hat{G} = -\delta(\mathbf{x}-\mathbf{x}^*)$$

Solution:

$$\hat{G}(\mathbf{x},\mathbf{x}',\omega) = -\frac{e^{i\frac{\omega}{c}|\mathbf{x}-\mathbf{x}^*|}}{4\pi\,|\,\mathbf{x}-\mathbf{x}^*\,|}$$

This is the acoustic wave generated by a balloon at **x***, pulsating with frequency ω.


wavelength: $\lambda = 2\pi c/\omega$
wavefronts
x_3
λ
x^*
x_2
0
pulsating
ballon
x_1
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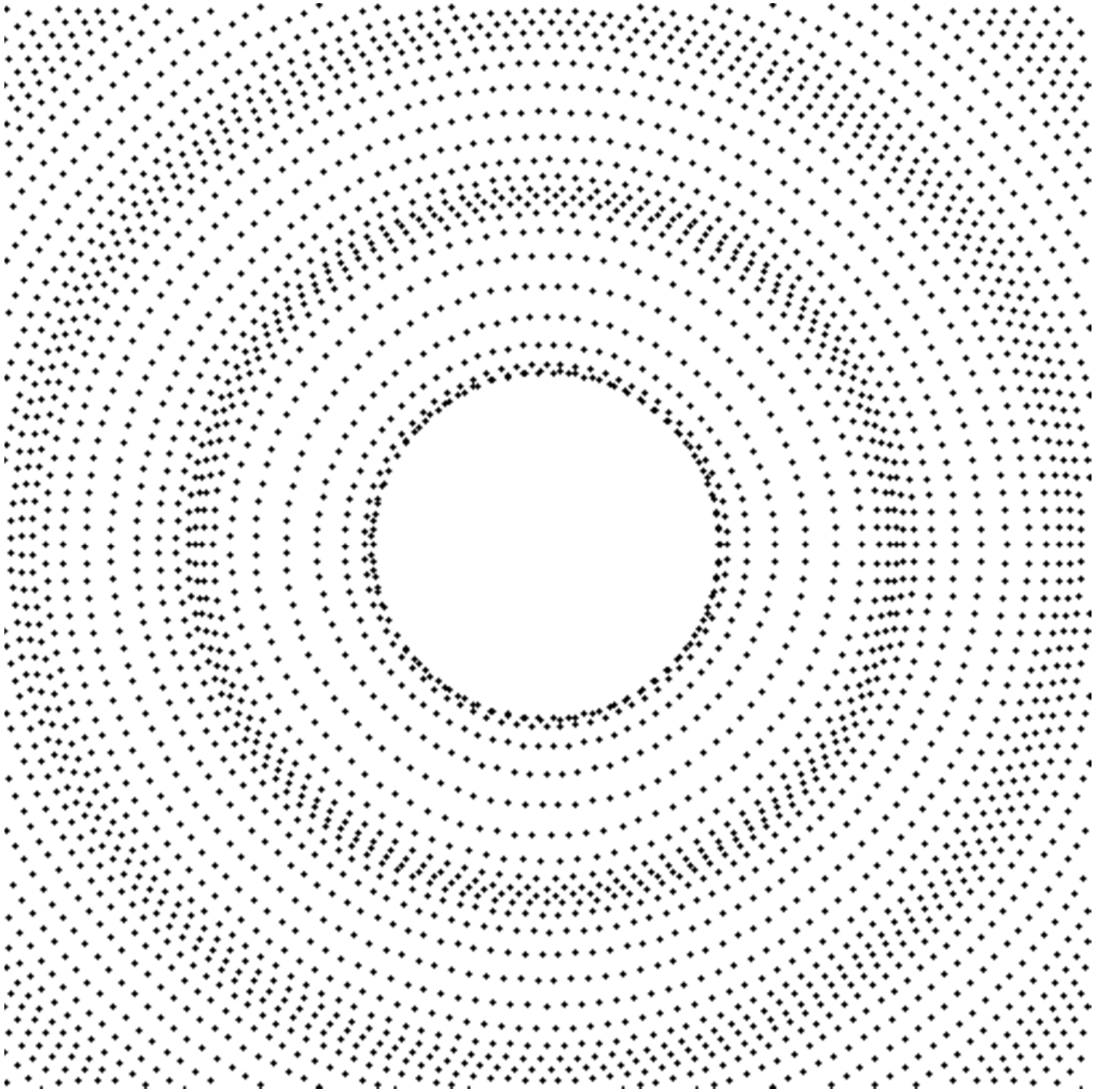
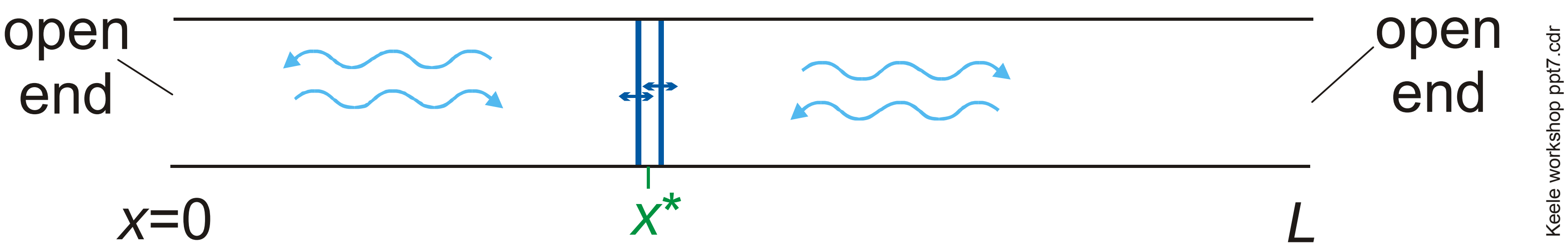


(animation: O.A. Power)

3.2. Tailored Green's function

Idea: extend the concept of the frequency-domain Green's function to fluids with boundaries

Example: 1-D tube with open ends



Governing equations:

PDE: $$\left(\frac{\omega}{c}\right)^2 \hat{G}(x,x^*,\omega) + \frac{\partial^2 \hat{G}}{\partial x^2} = -\delta(x - x^*)$$

bc's: $$\hat{G}(x,x^*,\omega)\Big|_{x=0} = 0\,, \quad \hat{G}(x,x^*,\omega)\Big|_{x=L} = 0$$

To find a solution, assume a superposition of backward and forward travelling waves,

$$\hat{G}(x,x^*,\omega)=\begin{cases} A_+ \mathrm{e}^{ikx}+A\_\mathrm{e}^{-ikx} & \text{for} \quad x<x^* \\ B_+ \mathrm{e}^{ikx}+B\_\mathrm{e}^{-ikx} & \text{for} \quad x>x^* \end{cases} \qquad \text{with } k=\tfrac{\omega}{c}$$

A_+, A_-, B_+, B_- : amplitudes (unknown)

4 equations required: 2 from bc's
2 from PDE

calculation (not shown)

Result:

$$\hat{G}(x,x^*,\omega)=\begin{cases} \dfrac{1}{\frac{\omega}{c}\sin\frac{\omega L}{c}}\sin\dfrac{\omega x}{c}\sin\dfrac{\omega(x^*-L)}{c} & \text{for} \quad x<x^* \\ \dfrac{1}{\frac{\omega}{c}\sin\frac{\omega L}{c}}\sin\dfrac{\omega(x-L)}{c}\sin\dfrac{\omega x^*}{c} & \text{for} \quad x>x^* \end{cases}$$

Summary

The frequency-domain Green’s function is the response to a harmonic point source (1-D, 2-D and 3-D).

Its governing equation is the Helmholtz equation with forcing term – δ ($\mathbf{x}$-$\mathbf{x}$*).

There is a free-space version and a tailored version.

4. Compact Green's function

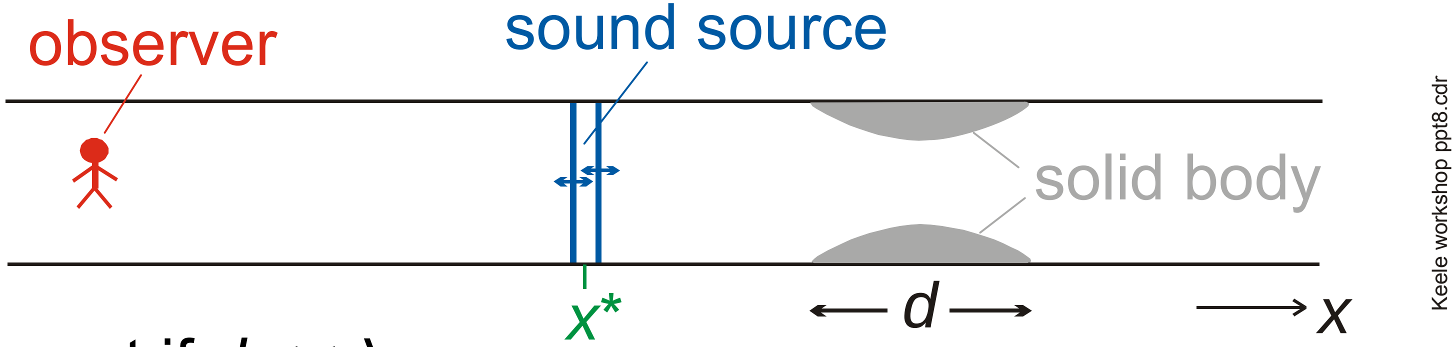
also called "low-frequency Greens function"

4.1. General concept

The compact Green's function is an approximation of the tailored Green's function, which is valid if the following conditions are satisfied:

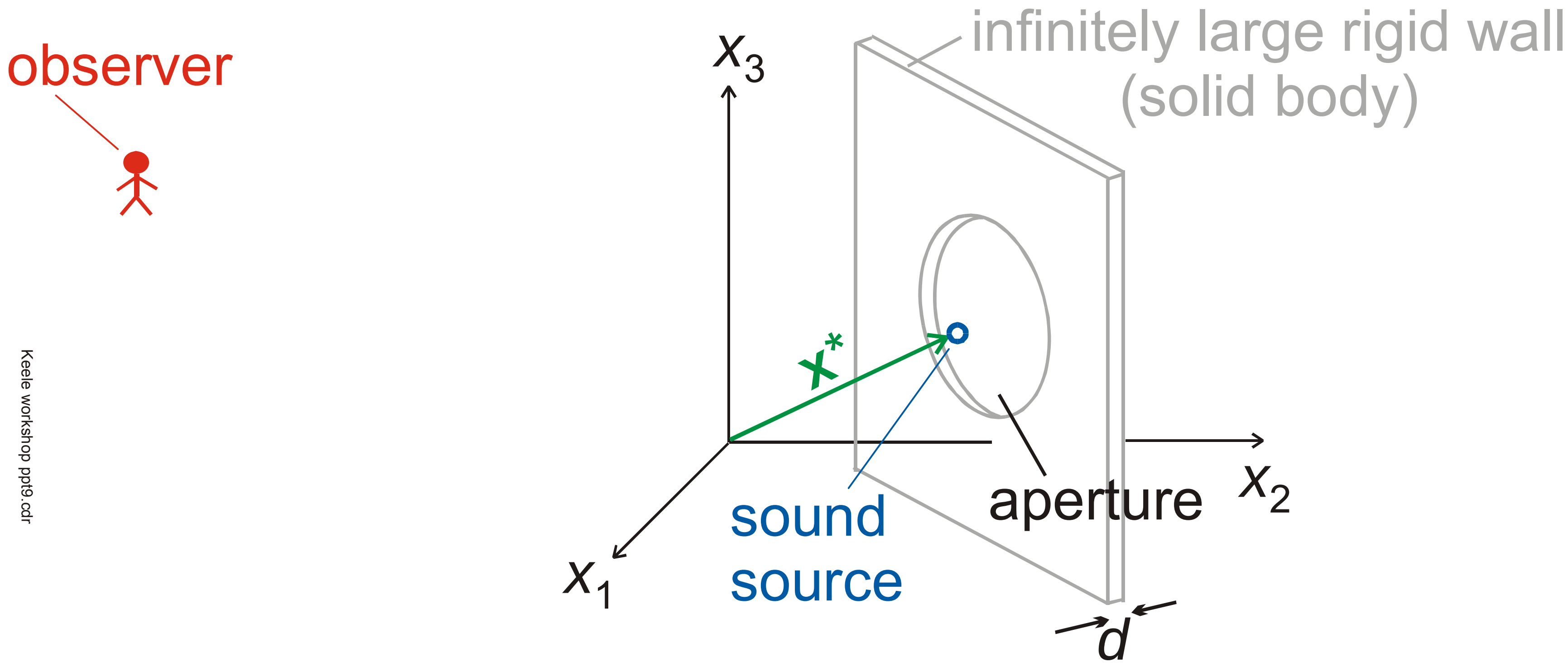
- the region of interest contains a compact solid body
- the sound source is close to the solid body, and the observer far away from it.

Example (1) Tube with localised constriction



compact if $d \ll \lambda$

Example (2) Orifice plate



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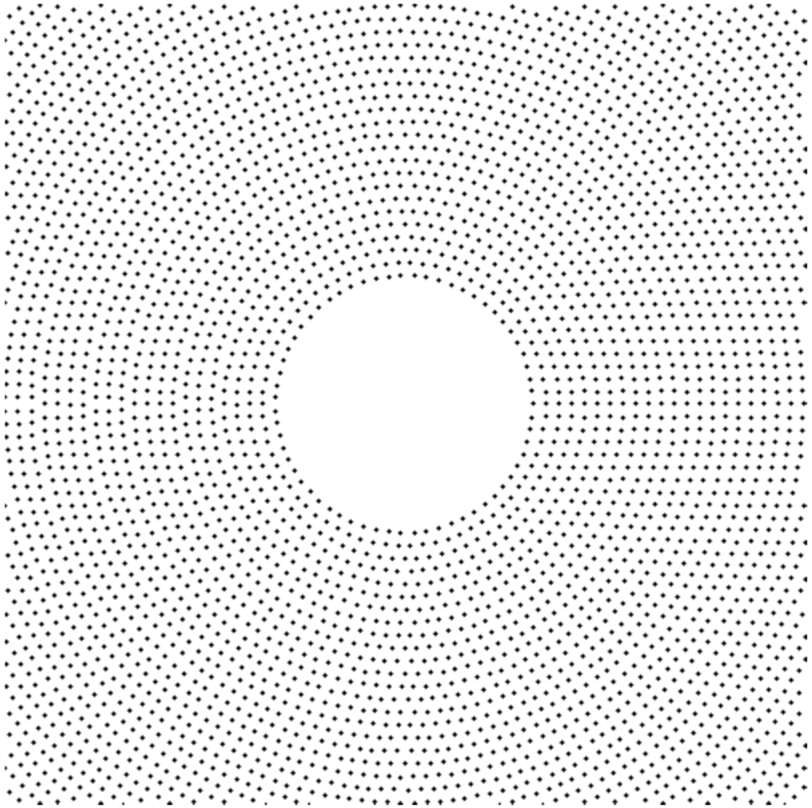
compact if $d \ll \lambda$ and radius $\ll \lambda$

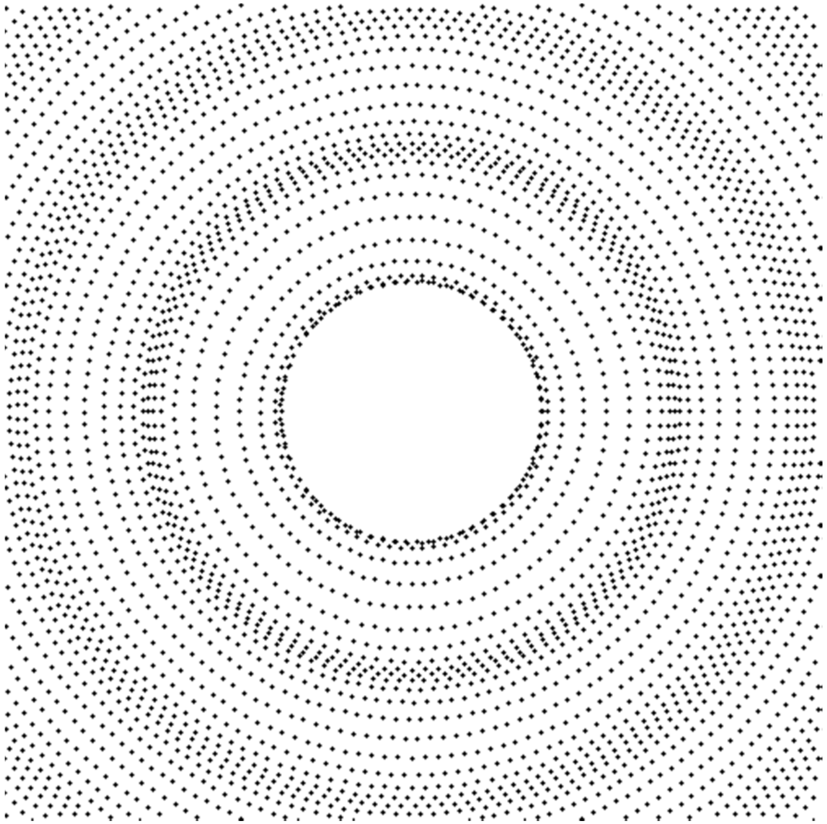
For compact geometries, the fluid near the solid body behaves like an incompressible fluid.

comparison: compact – non-compact

$d << \lambda$



$d > \lambda$



(animations: O.A. Power)

4.2. Calculation method

incompressible fluid: ρ=const and $c \to \infty$

The governing equation for the Green's function simplifies.

time-domain:

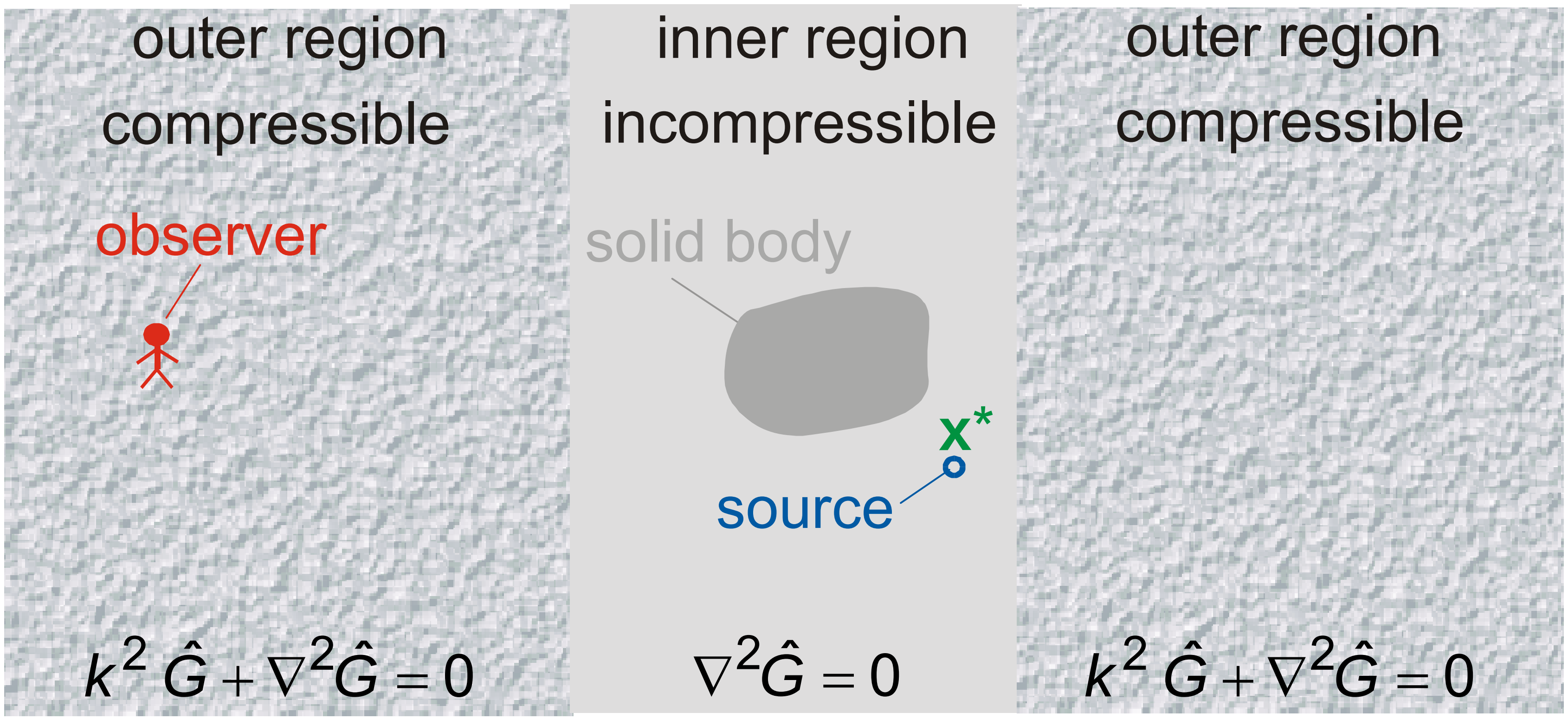
$$\underbrace{\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}}_{\to 0 \text{ as } c \to \infty} - \nabla^2 G = \delta(\mathbf{x}-\mathbf{x}^*)\delta(t-t^*)$$

frequency-domain:

$$\underbrace{(\tfrac{\omega}{c})^2 \hat{G}(\mathbf{x},\mathbf{x}',\omega)}_{\to 0 \text{ as } c \to \infty} + \nabla^2 \hat{G} = -\delta(\mathbf{x}-\mathbf{x}^*)$$

General calculation method

divide into compressible and incompressible regions:



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Asymptotic matching

Step 1: Solve in inner region

zero normal velocity on surface of solid body

$$\nabla^2 \hat{G} = 0 \quad \text{with} \quad \frac{\partial \hat{G}}{\partial n} = 0$$

$\rightarrow$ inner solution $\hat{G}^{(i)}$

Step 2: Solve in outer region

$$k^2 \hat{G} + \nabla^2 \hat{G} = 0 \,, \qquad \text{ignoring solid body}$$

$\rightarrow$ outer solution $\hat{G}^{(o)}$

Step 3: Assume that there is an intermediate region, where both inner and outer solutions are valid, and match the two solutions.

Applications of the compact Green's function

orifice plates

tube ends

tubes with blockage

Thank you!

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