Applications of the Green's function in thermoacoustics

Maria Heckl

Department of Mathematics, Keele University, UK

e-mail: m.a.heckl@keele.ac.uk

1. Time history calculations

1.1. Green's function method

1.2. Comparison with the Galerkin method

2. Stability analysis for individual modes

1. Time history calculations

1.1. Green's function method

acoustic analogy equation for the velocity potential Φ

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=\underbrace{-\frac{\gamma-1}{c^2}q'(x,t)}_{\text{forcing term}}$$

fluctuations of rate of heat release (per unit mass)

alternative form for the acoustic pressure *p*'

$$\frac{1}{c^2}\frac{\partial^2 p'}{\partial t^2}-\frac{\partial^2 p'}{\partial x^2}=\underbrace{\frac{\gamma-1}{c^2}\bar{\rho}\frac{\partial q'}{\partial t}(x,t)}_{\text{forcing term}}$$

Forced PDEs → suitable for Green's function approach

Governing equation for the Green's function

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}-\frac{\partial^2 G}{\partial x^2}=\delta(x-x^*)\delta(t-t^*)$$

Combine with equation for Φ

$$\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}-\frac{\partial^2 \phi}{\partial x^2}=-\frac{\gamma-1}{c^2}\;q'(x,t)$$

to get integral equation

$$\phi(x,t)=-\frac{\gamma-1}{c^2}\int_{t^*=0}^{t}\underbrace{\int_{x^*=-\infty}^{\infty} G(x,x^*,t-t^*)\,q'(x^*,t^*)\,\mathrm{d}x^*}\,\mathrm{d}t^*$$

can be calculated for a compact heat source at x_q,

$$q'(x,t)=q(t)\,\delta(x-x_q)$$

$$\phi(x,t) = -\frac{\gamma-1}{c^2}\int_{t^*=0}^{t} G(x,x_q,t-t^*)\,q(t^*)\,\mathrm{d}t^* \quad \Bigg| \quad \frac{\partial}{\partial x}, \quad \text{evaluate at } x = x_q$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=x_q} \qquad \qquad \left.\frac{\partial G}{\partial x}\right|_{\substack{x=x_q\\x^*=x_q}}$$

equation for velocity at x_q :

$$u_q(t) = -\frac{\gamma-1}{c^2}\int_{t^*=0}^{t} \left.\frac{\partial G}{\partial x}\right|_{\substack{x=x_q\\x^*=x_q}} q(t^*)\,\mathrm{d}t^*$$

Assume time-lag law: $q(t) = q(u_q(t-\tau))$ (linear or nonlinear)

$$u_q(t) = -\frac{\gamma-1}{c^2}\int_{t^*=0}^{t} \left.\frac{\partial G}{\partial x}\right|_{\substack{x=x_q\\x^*=x_q}} q(u_q(t^*-\tau))\,\mathrm{d}t^*$$

velocity at current time

velocity at earlier time

This is an integral equation for $u_q(t)$ (Volterra equation).

Solve with iteration: time-stepping method

Discretise: $t \rightarrow t_m = 0, \Delta t, 2\Delta t, \ldots\ m\Delta t$

$$t^* \rightarrow t_j^* = 0, \Delta t, 2\Delta t, \ldots\ j\Delta t$$

$$\int_{t^*=0}^{t} \ldots \mathrm{d}t^* \rightarrow \sum_{j=1}^{m} \ldots \Delta t$$

Then

$$u_q(t_m) = \sum_{j=0}^{m} g(t_m - t_j^*)\ q(t_j^*)\ \Delta t$$

abbreviation for $-\frac{\gamma-1}{c^2}\frac{\partial G}{\partial x}\Big|_{\substack{x=x_q \\ x^*=x_q}}$

This can be solved iteratively.

Initial conditions: the initial heat pulse is known, $q(t)\big|_{t=0} = q_0$

the velocity before $t = 0$ is zero,

$$u_q(t-\tau) = 0 \quad \text{for} \quad t-\tau < 0$$

First few iteration steps:

$$m = 0, t_m = 0:$$

$$q(0) = q_0$$

$$u_q(0) = g(0)q(0)$$

$$m = 1, t_m = \Delta t:$$

$$q(\Delta t) = q(u_q(\Delta t - \tau))$$

$$u_q(\Delta t) = \underbrace{g(\Delta t)q(0)}_{j=0} + \underbrace{g(0)q(\Delta t)}_{j=1}$$

$$m = 2, t_m = 2\Delta t:$$

$$q(\Delta t) = q(u_q(2\Delta t - \tau))$$

$$u_q(2\Delta t) = \underbrace{g(2\Delta t)q(0)}_{j=0} + \underbrace{g(\Delta t)q(\Delta t)}_{j=1} + \underbrace{g(0)q(2\Delta t)}_{j=2}$$

.....

Problem: As *m* increases, more and more terms need to be calculated and added.

Idea: Find a more efficient iteration scheme by exploiting the fact that we know the Green's function analytically as a superposition of modes.

$$G(x, x^*, t - t^*) = \sum_{n=1}^{N} G_n(x, x^*)\, e^{-i\,\omega_n (t - t^*)} \quad \text{for} \quad t > t^*$$

ω_n: eigenfrequencies

G_n: Green’s function amplitudes

N: maximum number of modes considered

Introduce abbreviation $g_n = -\frac{\gamma - 1}{c^2} \frac{\partial G_n}{\partial x}\bigg|_{\substack{x = x_q \\ x^* = x_q}}$, then

$$u_q(t) = \int_{t^*=0}^{t} \sum_{n=1}^{N} g_n\, e^{-i\omega_n (t - t^*)} q(t^*) \mathrm{d}t^*$$

Collect terms with t^*:

$$u_q(t) = \sum_{n=1}^{N} g_n \, e^{-i\omega_n t} \underbrace{\int_{t^*=0}^{t} e^{i\omega_n t^*} q(t^*) dt^*}_{\mathcal{I}_n(t) \text{ (abbreviation)}}$$

split up integration range: $\int_{t^*=0}^{t} = \int_{t^*=0}^{t-\Delta t} + \int_{t^*=t-\Delta t}^{t}$

Then

$$\mathcal{I}_n(t) = \underbrace{\int_{t^*=0}^{t-\Delta t} e^{i\omega_n t^*} q(t^*) dt^*}_{= \mathcal{I}_n(t-\Delta t)} + \underbrace{\int_{t^*=t-\Delta t}^{t} e^{i\omega_n t^*} q(t^*) dt^*}_{\approx q(t) \underbrace{\int_{t^*=t-\Delta t}^{t} e^{i\omega_n t^*} dt^*}_{= \frac{e^{i\omega_n t}}{i\omega_n}\left(1 - e^{-i\omega_n \Delta t}\right)}}$$

Iteration scheme

$$u_q(t) = \sum_{n=1}^{N} g_n \, e^{-i\omega_n t} \, \mathcal{I}_n(t)$$

with

$$\mathcal{I}_n(t) = \mathcal{I}_n(t-\Delta t) + q(t)\frac{e^{i\omega_n t}}{i\omega_n}\left(1 - e^{-i\omega_n \Delta t}\right)$$

$$q(t) = q(u_q(t-\tau))$$

Only N terms need to be calculated in each iteration step.

Example: **Rijke tube**

ideal open ends: $\phi(0,t) = 0, \quad \phi(L,t) = 0$

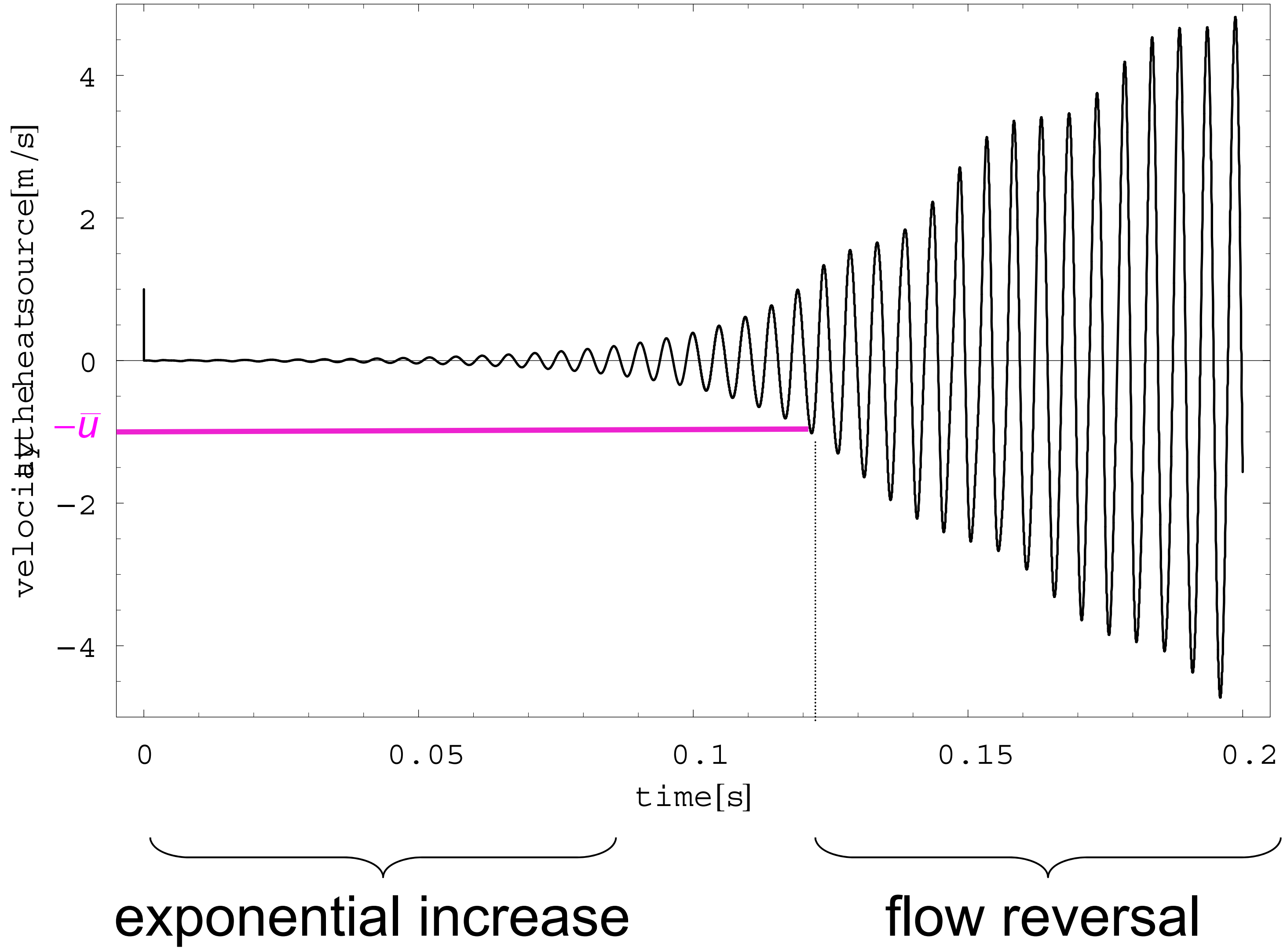
hot-wire gauze at x_q

heat release characteristic from hot wire theory:

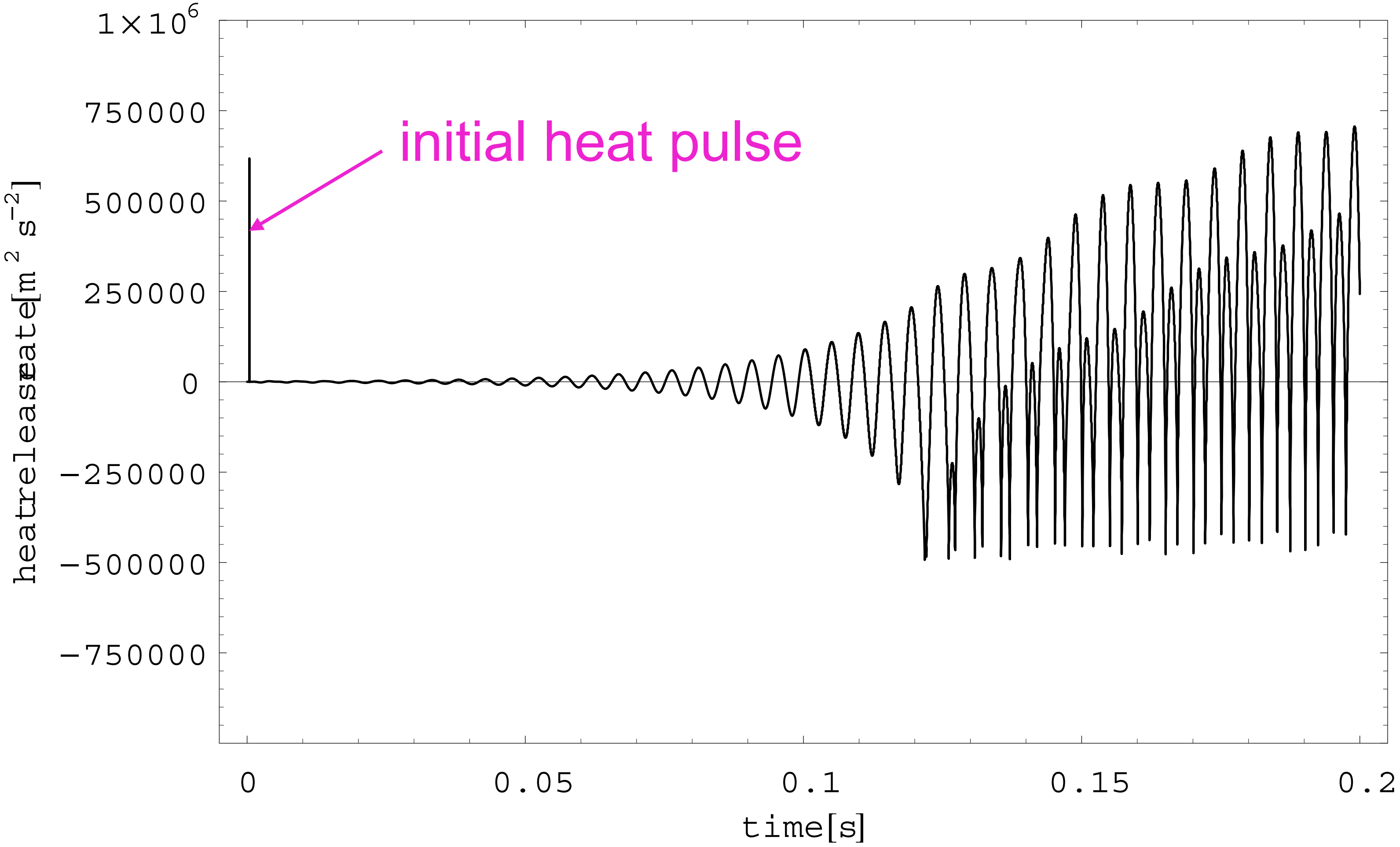
$$q(t) = a + b\sqrt{\left|\bar{u} + u_q(t-\tau)\right|}$$ **nonlinear!**

(a, b: constants; $\bar{u}$: mean flow velocity)

Time history of the velocity



Time history of the heat release rate (fluctuating part only)



Summary

If we know the

- tailored Green's function
- heat release law

of a thermoacoustic system, then we can calculate the time histories

- $u_q(t)$ (velocity at the heat source)
- $q(t)$ (heat release rate)

from a straightforward iteration scheme, stepping forward in time.

1.2. Comparison with Galerkin method

Idea: expand the acoustic field in terms of idealised eigenmodes

pressure: $$p'(x,t) = \sum_{n=1}^{N} p_n(t)\sin(n\pi x)$$

velocity: $$u'(x,t) = \sum_{n=1}^{N} u_n(t)\cos(n\pi x)$$

for tube with open ends

These are normalised: $\omega_n = n\pi$

practical tubes have: end correction
temperature gradient
change in cross-sectional area
…

Note: The Galerkin modes are an *approximation* of the real modes in the tube.

substitute into conservation equations

mass: $$\frac{\partial p'}{\partial t} + \gamma M \frac{\partial u'}{\partial x} = (\gamma - 1) q(t) \delta(x - x_q)$$

$$\sum_{n=1}^{N} \left[\dot{p}_n(t) - \gamma M n \pi u_n(t) \right] \sin(n\pi x) = (\gamma - 1) q(t) \delta(x - x_q)$$

orthogonality of sin(*nπx*) gives (see aside)

$$\dot{p}_n(t) - \gamma M n \pi u_n(t) = (\gamma - 1) q(t) \sin(n\pi x_q)$$

momentum: $$\gamma M \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial t} = 0$$

$$\sum_{n=1}^{N} \left[\gamma M \dot{u}_n(t) + n\pi p_n(t) \right] \cos(n\pi x) = 0$$

orthogonality of cos(*nπx*) gives

$$\dot{u}_n(t) + \frac{n\pi}{\gamma M} p_n(t) = 0$$

Aside

orthogonality of the eigenfunctions:

$$\int_{x=0}^{1} \sin(n\pi x)\sin(n'\pi x)\mathrm{d}x = \begin{cases} 0 & \text{if } n \neq n' \\ \frac{1}{2} & \text{if } n = n' \end{cases} \quad \text{(same for } \cos(n\pi x)\text{)}$$

This will help to separate the modes, e.g. in mass equation:

$$\sum_{n=1}^{N} \left[\dot{p}_n(t) - \gamma M n\pi u_n(t)\right]\sin(n\pi x) = (\gamma-1)q(t)\delta(x-x_q)$$

multiply both sides by sin(n'πx), integrate over tube $\int_{x=0}^{1} \ldots \mathrm{d}x$

$$\sum_{n=1}^{N} \left[\dot{p}_n(t) - \gamma M n\pi u_n(t)\right] \overbrace{\int_{x=0}^{1} \sin(n\pi x)\sin(n'\pi x)\mathrm{d}x}^{=0,\text{ unless } n = n'} = (\gamma-1)q(t)\sin(n'\pi x_q)$$

Only the term n = n' remains,

$$\dot{p}_{n'}(t) - \gamma M n'\pi u_{n'}(t) = (\gamma-1)q(t)\sin(n'\pi x_q)$$

From conservation equations:

$$\dot{u}_n(t) + \frac{n\pi}{\gamma M} p_n(t) = 0$$

$$\dot{p}_n(t) - \gamma M n\pi u_n(t) = (\gamma - 1) q(t) \sin(n\pi x_q)$$

in matrix form:

$$\underbrace{\begin{bmatrix} \dot{u}_1 \\ \dot{p}_1 \\ \dot{u}_2 \\ \dot{p}_2 \\ \vdots \end{bmatrix}}_{\dot{\Psi}(t)} = \underbrace{\begin{bmatrix} 0 & \frac{-\pi}{\gamma M} & 0 & 0 & \cdots \\ \frac{\pi}{\gamma M} & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \frac{-2\pi}{\gamma M} & \cdots \\ 0 & 0 & \frac{2\pi}{\gamma M} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} u_1 \\ p_1 \\ u_2 \\ p_2 \\ \vdots \end{bmatrix}}_{\Psi(t)} + q(t) \underbrace{\begin{bmatrix} 0 \\ (\gamma - 1)\sin \pi x_q \\ 0 \\ (\gamma - 1)\sin \pi x_q \\ \vdots \end{bmatrix}}_{\mathbf{F}}$$

or

$$\dot{\Psi}(t) = \mathbf{M}\,\Psi(t) + q(t)\,\mathbf{F}$$

Matrix differential equation for $u_1(t), u_2(t), \ldots, p_1(t), p_2(t), \ldots$

Solution

Iteration by time stepping method

discretise: $t = 0, \Delta t, 2\Delta t, \ldots$

$$\dot{\Psi}(t) \approx \frac{\Psi(t+\Delta t) - \Psi(t)}{\Delta t}$$

The resulting equation

unit matrix

$$\Psi(t+\Delta t) = (\mathbf{M}\,\Delta t + \mathbf{I}\,)\Psi(t) + q(t)\,\mathbf{F}$$

with

$$q(t) = q(\sum_{n=1}^{N} u_n(t-\tau)\cos(n\pi x_q))$$

can be solved iteratively.

Initial conditions:

the initial heat pulse is known, $q(t)\big|_{t=0} = q_0$.

the amplitudes $u_n(0)$ and $p_n(0)$ are zero for all modes.

Comparison with the Green's function method

Both methods apply to compact heat sources and lead to explicit iteration schemes stepping forward in time.

Green's function modes: real physical modes, with

ω_n: resonance frequencies

$G_n(x,x^*)$: spatial distribution of modal amplitudes

truncation of sum: $\sum_{n=1}^{\infty} \rightarrow \sum_{n=1}^{N}$

N: largest relevant mode number, given by observation

Galerkin modes: approximation of physical modes, with

$\omega_n = n\pi$: resonance frequencies of ideal tube

$\sin(n\pi x)$: approximate distribution of modal amplitudes

truncation of sum: $\sum_{n=1}^{\infty} \rightarrow \sum_{n=1}^{N}$ - what is N ?

2. Stability analysis of individual modes

Integral equation for the velocity

$$u_q(t) = \mathrm{Re} \int_{t^*=0}^{t} \sum_{n=1}^{N} g_n \, \mathrm{e}^{-\mathrm{i}\omega_n (t-t^*)} q(t^*) \mathrm{d}t^*$$

omitted in earlier material

$$\mathrm{Re}[\ldots] = \tfrac{1}{2}\left([\ldots] + \overline{[\ldots]}\right)$$

complex conjugate

Assume that only mode *n* is present:

$$u_n(t) = \underbrace{\frac{1}{2} \int_{t^*=0}^{t} g_n \, \mathrm{e}^{-\mathrm{i}\omega_n (t-t^*)} q(t^*) \mathrm{d}t^*}_{\mathcal{I}_n(t)} + \underbrace{\frac{1}{2} \int_{t^*=0}^{t} \overline{g_n} \, \mathrm{e}^{\mathrm{i}\overline{\omega_n} (t-t^*)} q(t^*) \mathrm{d}t^*}_{\overline{\mathcal{I}_n(t)}}$$

or

$$u_n(t) = \tfrac{1}{2}\left[\mathcal{I}_n(t) + \overline{\mathcal{I}_n(t)}\right] \quad \text{with} \quad \mathcal{I}_n(t) = \int_{t^*=0}^{t} g_n \, \mathrm{e}^{-\mathrm{i}\omega_n (t-t^*)} q(t^*) \mathrm{d}t^*$$

Conversion of integral equation into differential equation

Step 1: Determine $\frac{\partial \mathcal{I}_n}{\partial t}$, noting that *t* appears in the integrand *and* in the integration limit. $\frac{\partial \mathcal{I}_n}{\partial t} = -\mathrm{i}\omega_n \mathcal{I}_n + g_n q(t)$

Step 2: Use this result to calculate $\dot{u}_n$ and $\ddot{u}_n$.

$$\dot{u}_n = \tfrac{1}{2}\left[(g_n + \overline{g_n})q(t) - \mathrm{i}\omega_n \mathcal{I}_n + \mathrm{i}\overline{\omega_n}\,\overline{\mathcal{I}_n}\right]$$

$$\ddot{u}_n = \tfrac{1}{2}\left[(g_n + \overline{g_n})\dot{q}(t) - \left(\mathrm{i}\omega_n g_n + \mathrm{i}\overline{\omega_n}\,\overline{g_n}\right)q(t) - \omega_n^2 \mathcal{I}_n - \overline{\omega_n}^2\,\overline{\mathcal{I}_n}\right]$$

Step 3: Multiply as indicated and add the resulting equations.

$$u_n(t) = \tfrac{1}{2}\left[\mathcal{I}_n(t) + \overline{\mathcal{I}_n(t)}\right] \quad \Big| \quad \cdot(\omega_n \overline{\omega_n})$$

$$\dot{u}_n = \tfrac{1}{2}\left[(g_n + \overline{g_n})q(t) - \mathrm{i}\omega_n \mathcal{I}_n + \mathrm{i}\overline{\omega_n}\,\overline{\mathcal{I}_n}\right] \quad \Big| \quad \cdot(\mathrm{i}\omega_n)$$

This eliminates $\overline{\mathcal{I}_n}$ and gives an expression for $\mathcal{I}_n$,

$$\mathcal{I}_n = \frac{1}{\omega_n + \overline{\omega_n}}\left[2(\mathrm{i}u_n + \overline{\omega_n} u_n) - \mathrm{i}(g_n + \overline{g_n})q(t)\right]$$

Step 4: In the equation for $\ddot{u}_n$, substitute for $\mathcal{I}_n$, and simplify.

Result

$$\underbrace{\ddot{u}_n - 2\,\mathrm{Im}(\omega_n)\,\dot{u}_n + |\,\omega_n\,|^2\,u_n}_{\text{damped harmonic oscillator}} = \underbrace{-\,\mathrm{Im}(\omega_n \overline{g_n})\,q(t) + \mathrm{Re}(g_n)\,\dot{q}(t)}_{\text{forcing term}}$$

Assume heat release law $q(t) = [n_1 u(t-\tau) - n_0 u(t)]$

amplitude-dependent coefficients (n_1, τ, n_0)
can be obtained e.g. from FDF measurements

substitute into oscillator equation:

$$\ddot{u}_n + \underbrace{\left[-2\,\mathrm{Im}(\omega_n) + n_0\,\mathrm{Re}(g_n)\right]}_{=c_1}\dot{u}_n + \underbrace{\left[|\,\omega_n\,|^2 - n_0\,\mathrm{Im}(\omega_n \overline{g_n})\right]}_{=c_0} u_n =$$

$$= \underbrace{\left[-n_1\,\mathrm{Im}(\omega_n \overline{g_n})\right]}_{=b_0} u_n(t-\tau) + \underbrace{\left[n_1\,\mathrm{Re}(g_n)\right]}_{=b_1} \dot{u}_n(t-\tau)$$

$$\ddot{u}_n(t) + c_1\dot{u}_n(t) + c_0 u_n(t) = b_0 u_n(t-\tau) + b_1\dot{u}_n(t-\tau)$$

We look for steady limit cycle solutions: $u_n(t) = A\cos\Omega t$, $\Omega \approx \omega_n$

For *any* time-lag:

$$u_n(t-\tau) = A\cos\Omega(t-\tau) = (\cos\Omega\tau)\,u_n(t) - \frac{\sin\Omega\tau}{\Omega}\dot{u}_n(t)$$

$$\dot{u}_n(t-\tau) = -\Omega A\sin\Omega(t-\tau) = (\Omega\sin\Omega\tau)\,u_n(t) + (\cos\Omega\tau)\dot{u}_n(t)$$

ODE for $u_n(t)$:

$$\ddot{u}_n(t) + \underbrace{\left[c_1 + b_0\frac{\sin\Omega\tau}{\Omega} - b_1\cos\Omega\tau\right]}_{=a_1}\dot{u}_n(t) + \underbrace{\left[c_0 - b_0\cos\Omega\tau - b_1\Omega\sin\Omega\tau\right]}_{=a_0}u_n(t) = 0$$

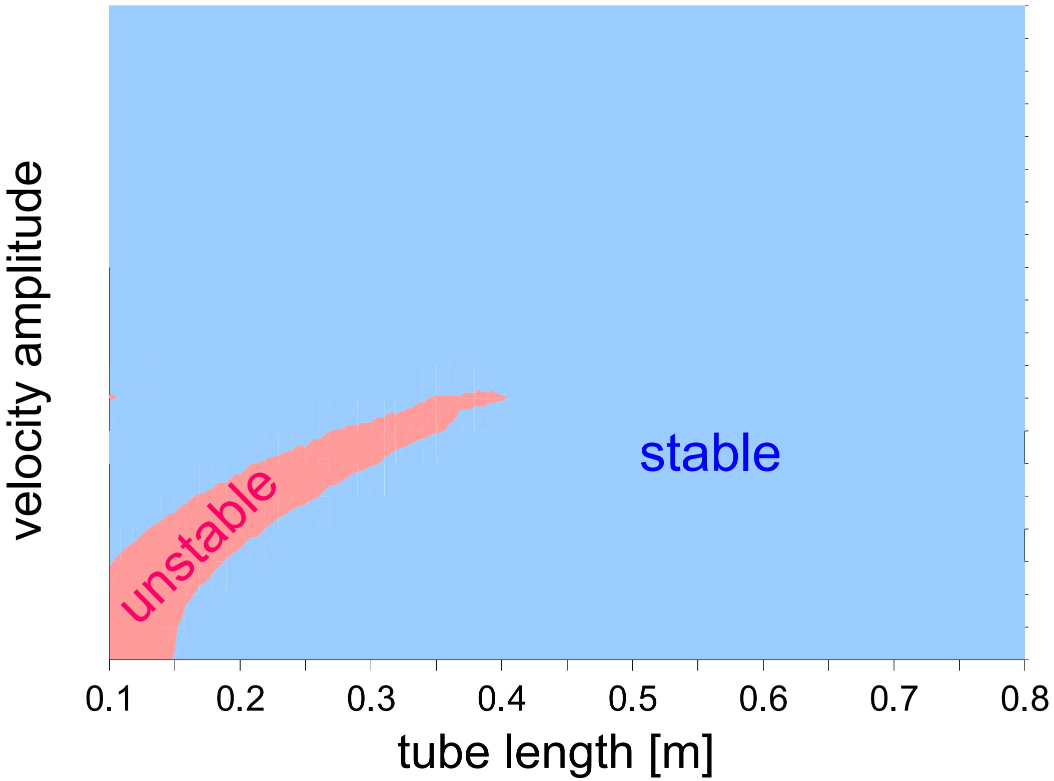
a_0 : oscillation frequency (squared)

a_1 : damping coefficient, amplitude-dependent

$a_1 > 0$: stability, $a_1 < 0$: instability

The stability behaviour can be examined at different amplitudes for various system parameters.

Example: ¼ wave resonator with variable length and amplitude-dependent time-lag law



time-lag: $\tau = \tau_0 + \tau_1 A^2$, A: velocity amplitude

Summary

Analysis works well for weakly coupled modes.

Straightforward stability predictions.

Suitable for cases where the nonlinear heat release law is given in terms of amplitude-dependent coefficients.

Thank you!

Maria Heckl
Department of Mathematics
Keele University
Staffordshire ST5 5BG
England
m.a.heckl@keele.ac.uk