

The Galerkin approach to 'reduce' models in thermoacoustics

Camilo F. Silva

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Even if LNSE or the Helmholtz equation are expressed in a state-space framework, they remain computationally expensive.

Linearized Navier Stokes Equations

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_j} (\bar{\rho} u'_j + \rho' \bar{u}_j) = 0$$

$$\frac{\partial}{\partial t} (\bar{\rho} u'_i + \rho' \bar{u}_i) + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{u}_i u'_j + \bar{\rho} u'_i \bar{u}_j + \rho' \bar{u}_i \bar{u}_j) = -\frac{\partial p'}{\partial x_i} + \frac{\partial \tau'_{ij}}{\partial x_j}$$

$$\bar{T} \left[\frac{\partial}{\partial t} (\bar{\rho} s' + \rho' \bar{s}) + \frac{\partial}{\partial x_j} (\bar{\rho} \bar{u}_j s' + \bar{\rho} u'_j \bar{s} + \rho' \bar{u}_j \bar{s}) \right] + T' \frac{\partial}{\partial x_j} (\bar{\rho} \bar{u}_j \bar{s}) = \dot{q}'$$

Helmholtz Equation

$$s^2 \hat{p} - \frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right) = s(\gamma - 1) \hat{q}$$

How to model systems that are complex in frameworks that are easy for computation?

Let us take a look at reduced order models

Outline

† Solving the Helmholtz Equation by modal expansion

† And the state space?

† About a one mode expansion

The Helmholtz Equation is treated as the reference model to approximate

$$s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = \frac{\hat{h}}{s(\gamma - 1) \hat{q}}$$

Requires a flame response

Helmholtz Equation

$$s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = \hat{h}$$

Boundary Conditions

$$\mathbf{n} \cdot \frac{\partial \hat{p}}{\partial x_i} = -\hat{f}$$

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An auxiliary problem is required for the approximation

Helmholtz Equation

$$s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L}\hat{p}} = \hat{h}$$

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Auxiliary Problem

$$s^2 \hat{G} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{G}}{\partial x_i} \right)}_{\mathcal{L}\hat{G}} = \delta(x - x_0)$$

$$\mathbf{n} \cdot \frac{\partial \hat{G}}{\partial x_i} = 0$$

The solution is given in terms of a Green's function G

Helmholtz Equation

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$$\mathbf{n} \cdot \frac{\partial \hat{G}}{\partial x_i} = 0$$

Solution

$$\hat{p} = \int_V G h dV + \int_{\partial V} G f dS$$

Let us solve the Auxiliary problem !

Helmholtz Equation

$$s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = \hat{h}$$

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Auxiliary Problem

$$s^2 \hat{G} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{G}}{\partial x_i} \right)}_{\mathcal{L} \hat{G}} = \delta(x - x_0) \quad \mathbf{n} \cdot \frac{\partial \hat{G}}{\partial x_i} = 0$$

Solution

$$\hat{p} = \int_V G h dV + \int_{\partial V} G f dS$$

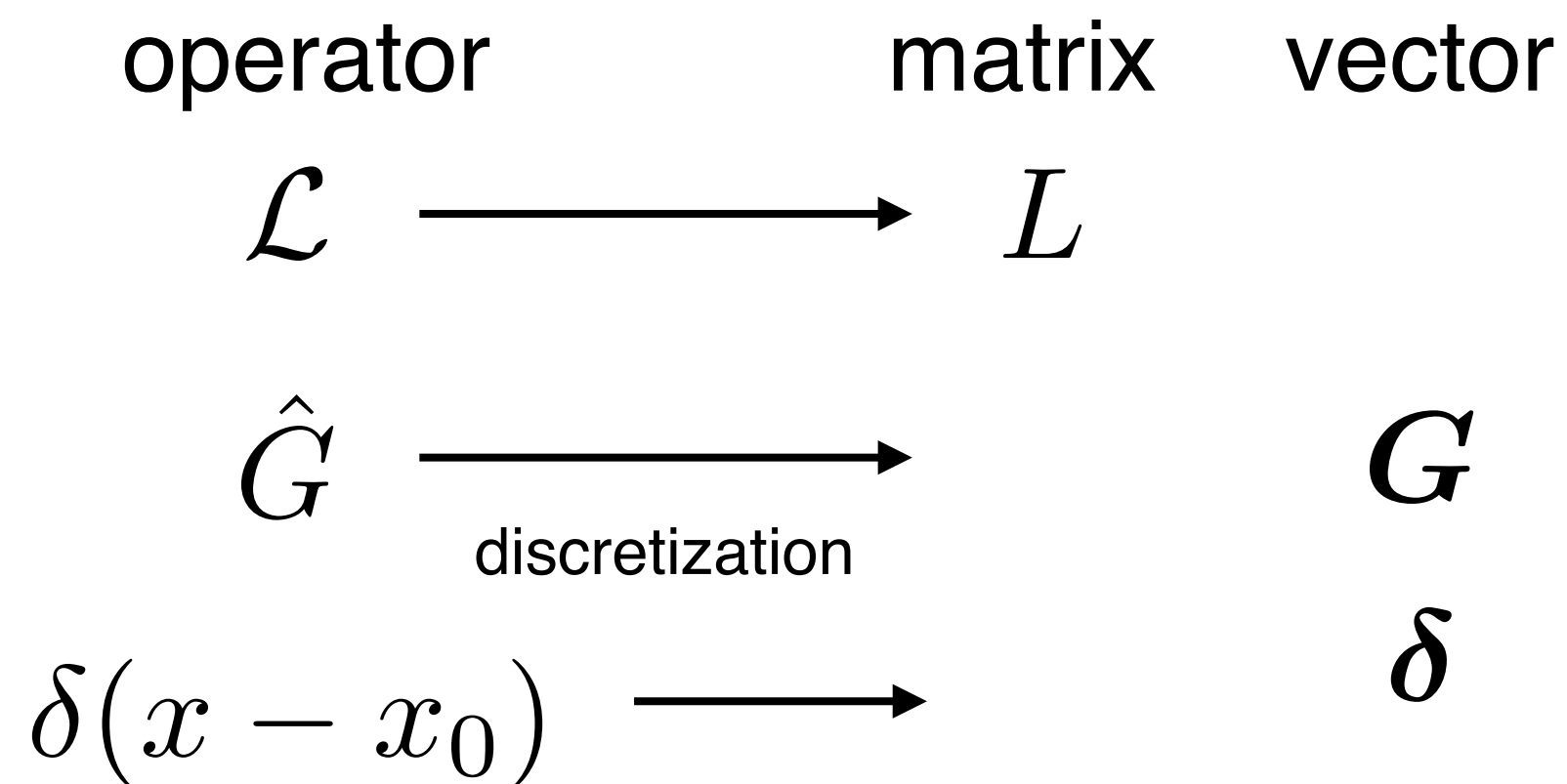
In the Helmholtz Equation, the matrix L gathers all the information of the system

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operator matrix vector
 \mathcal{L} \longrightarrow L

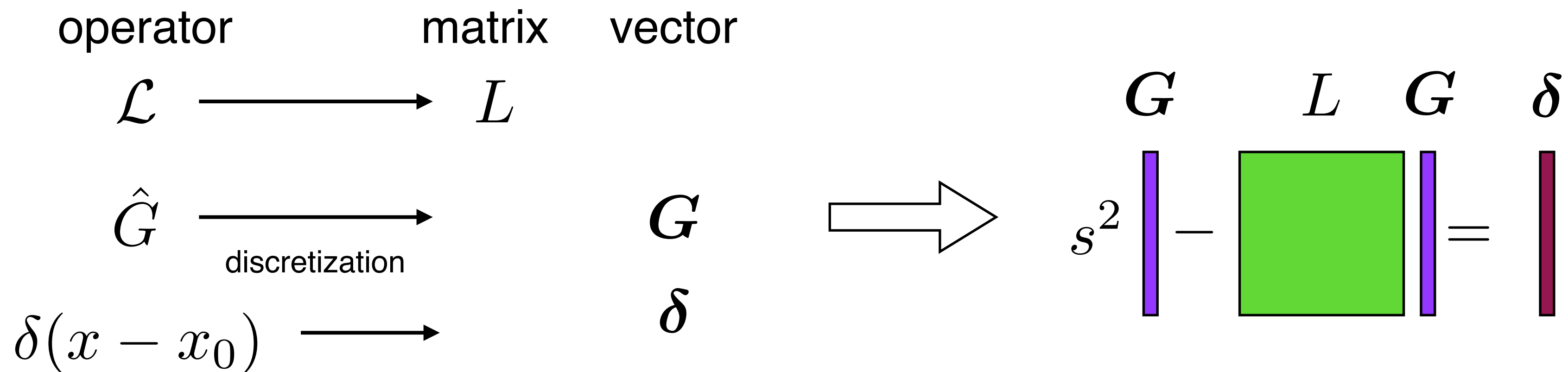
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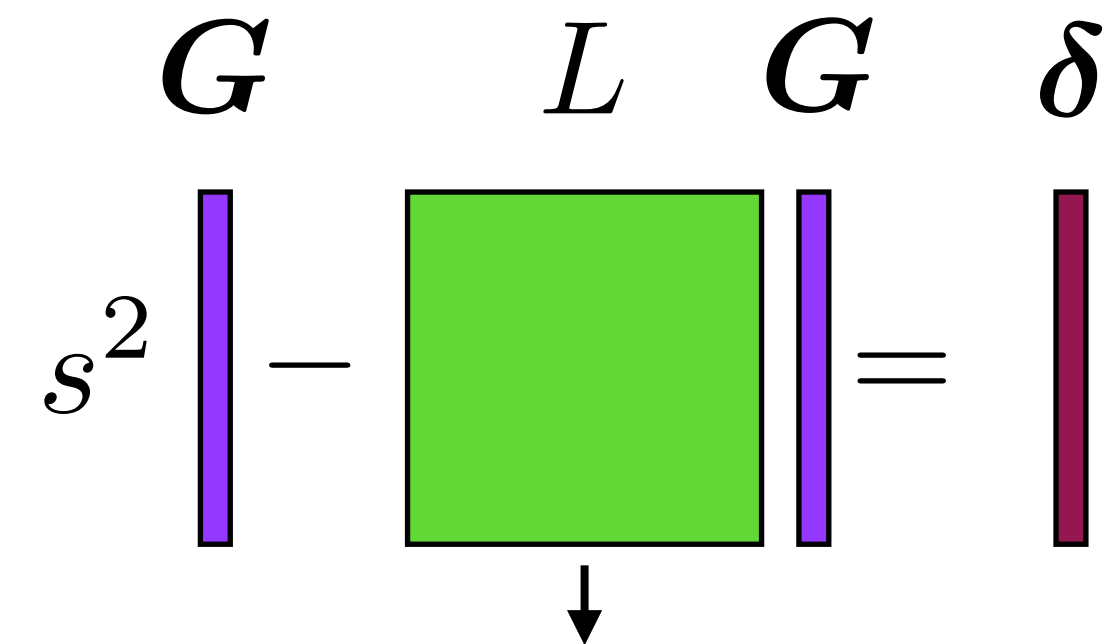
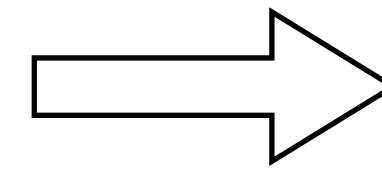
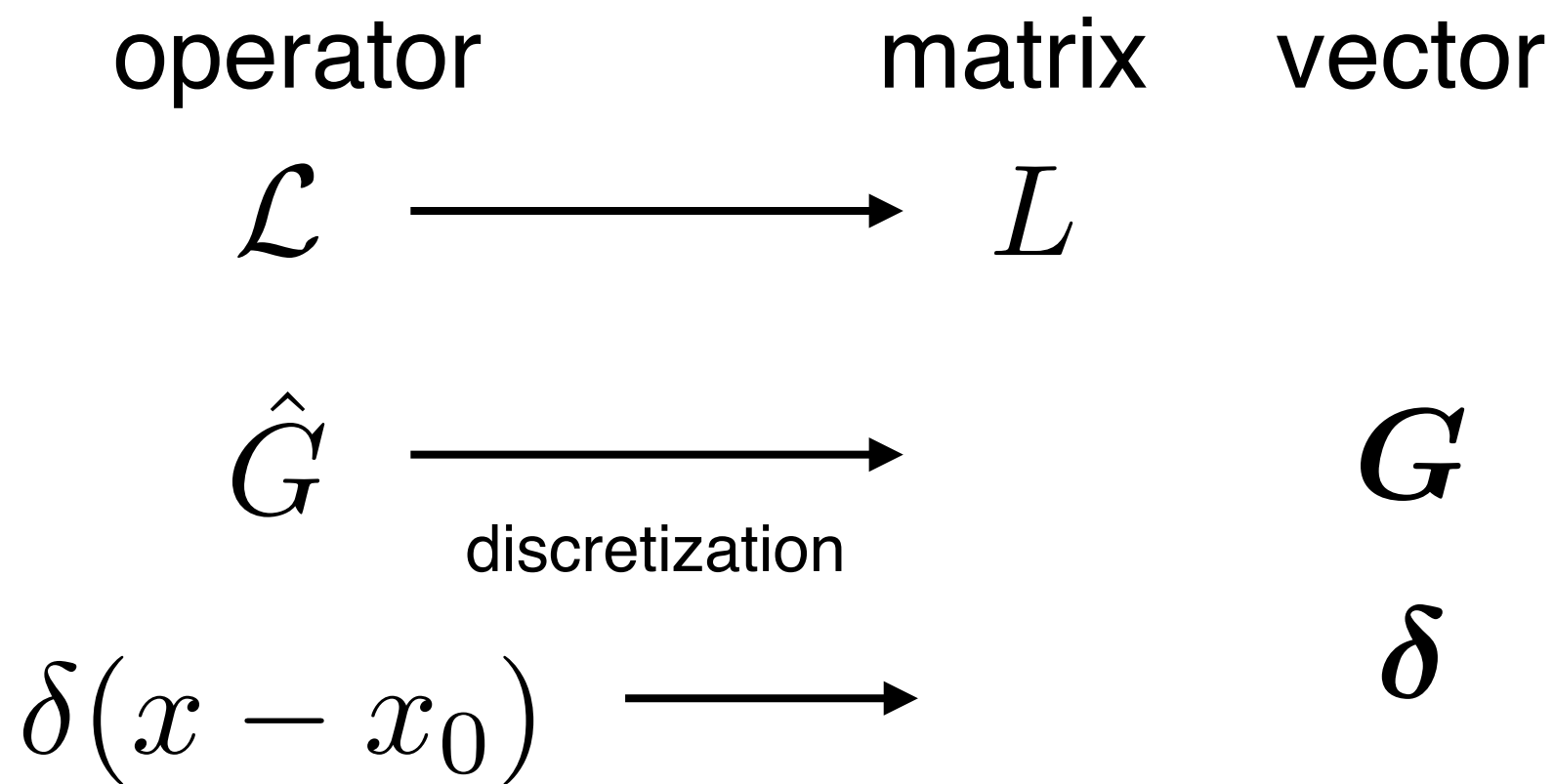
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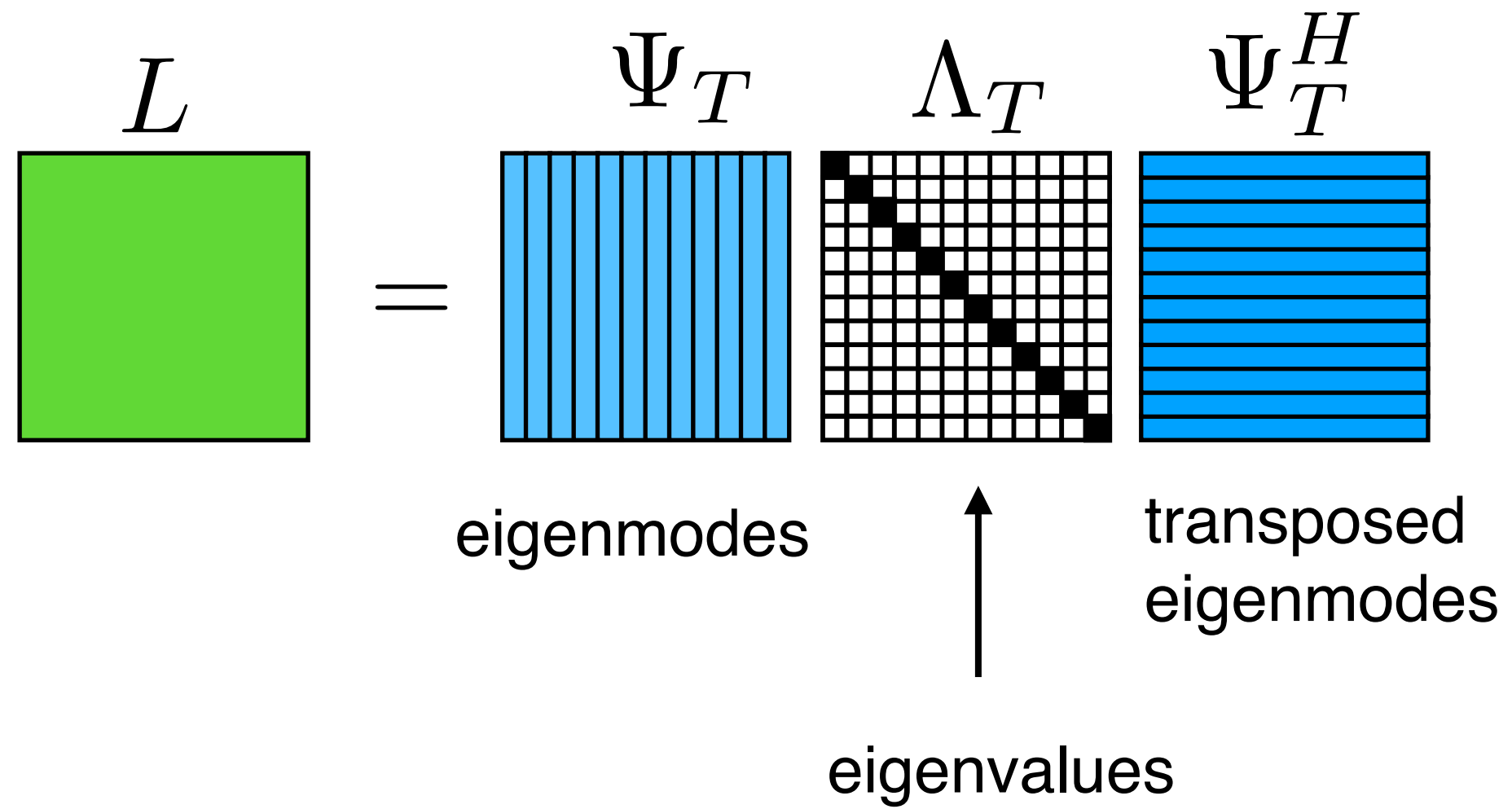
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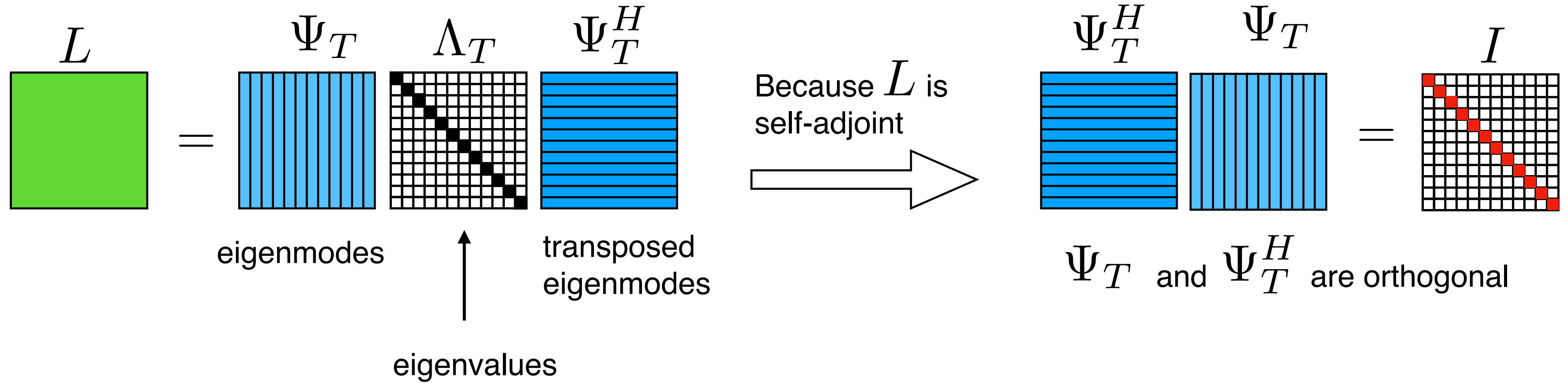


let us do an eigenvalue decomposition

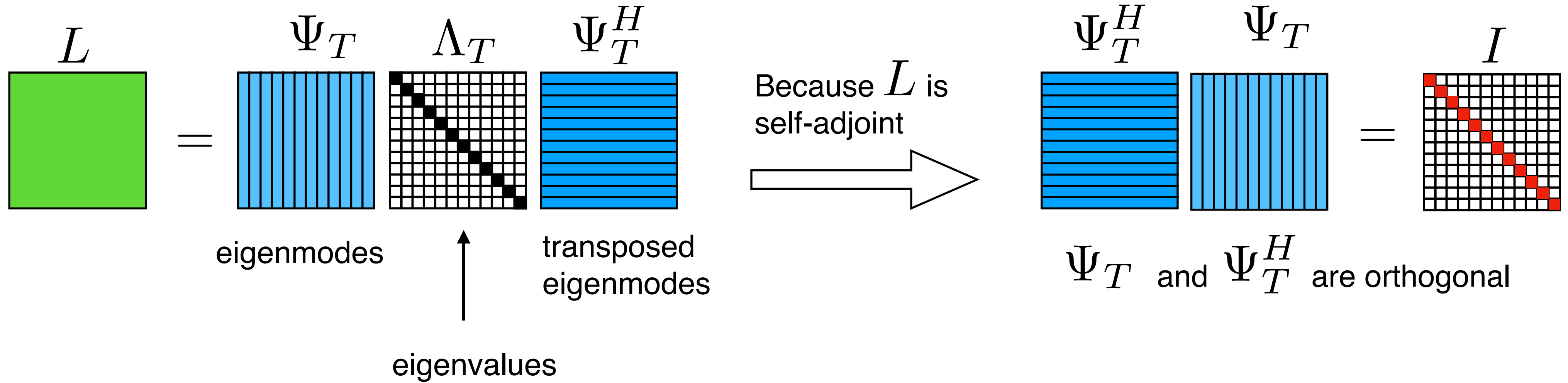
The auxiliary problem can be defined with ideal BC -> Self-adjoint Operator



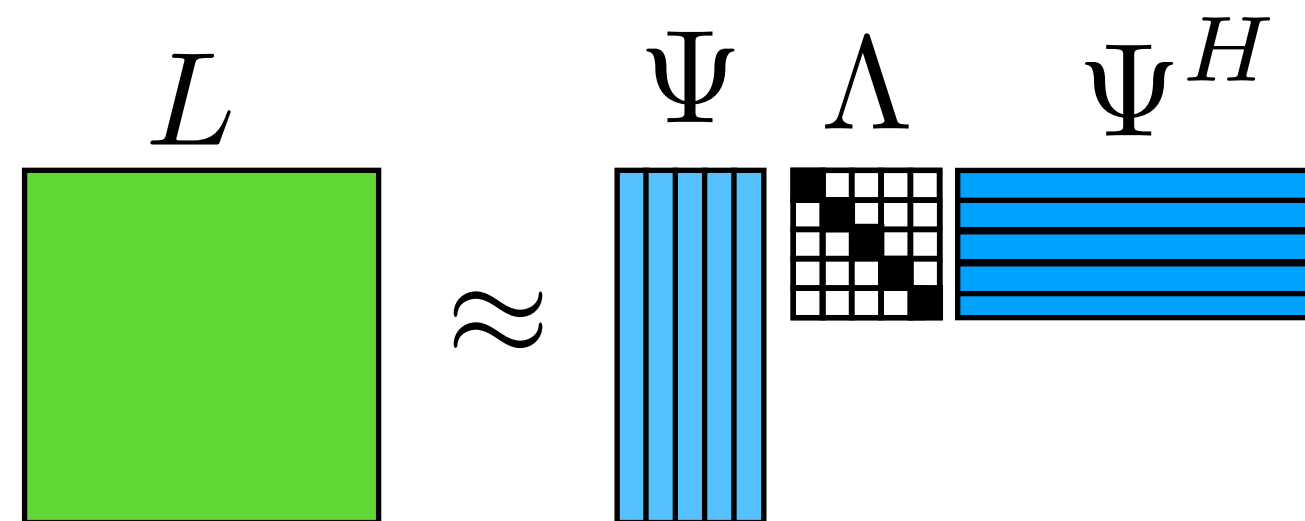
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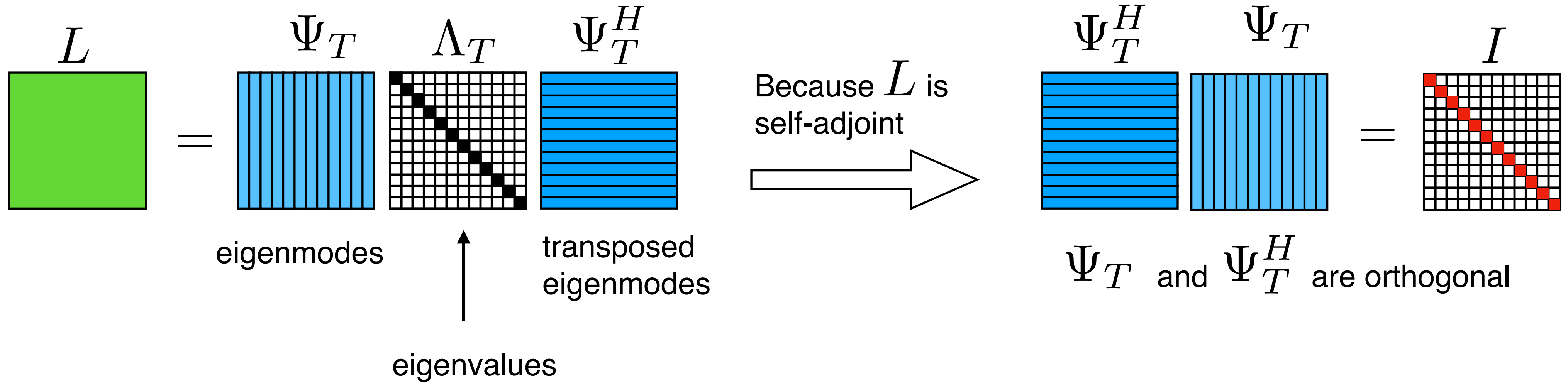
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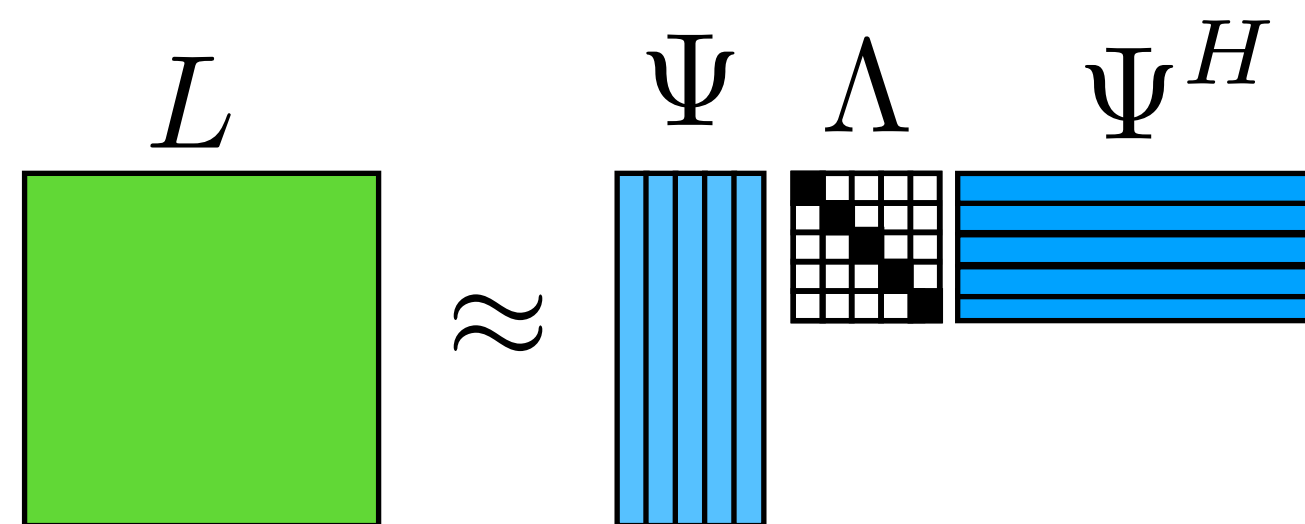
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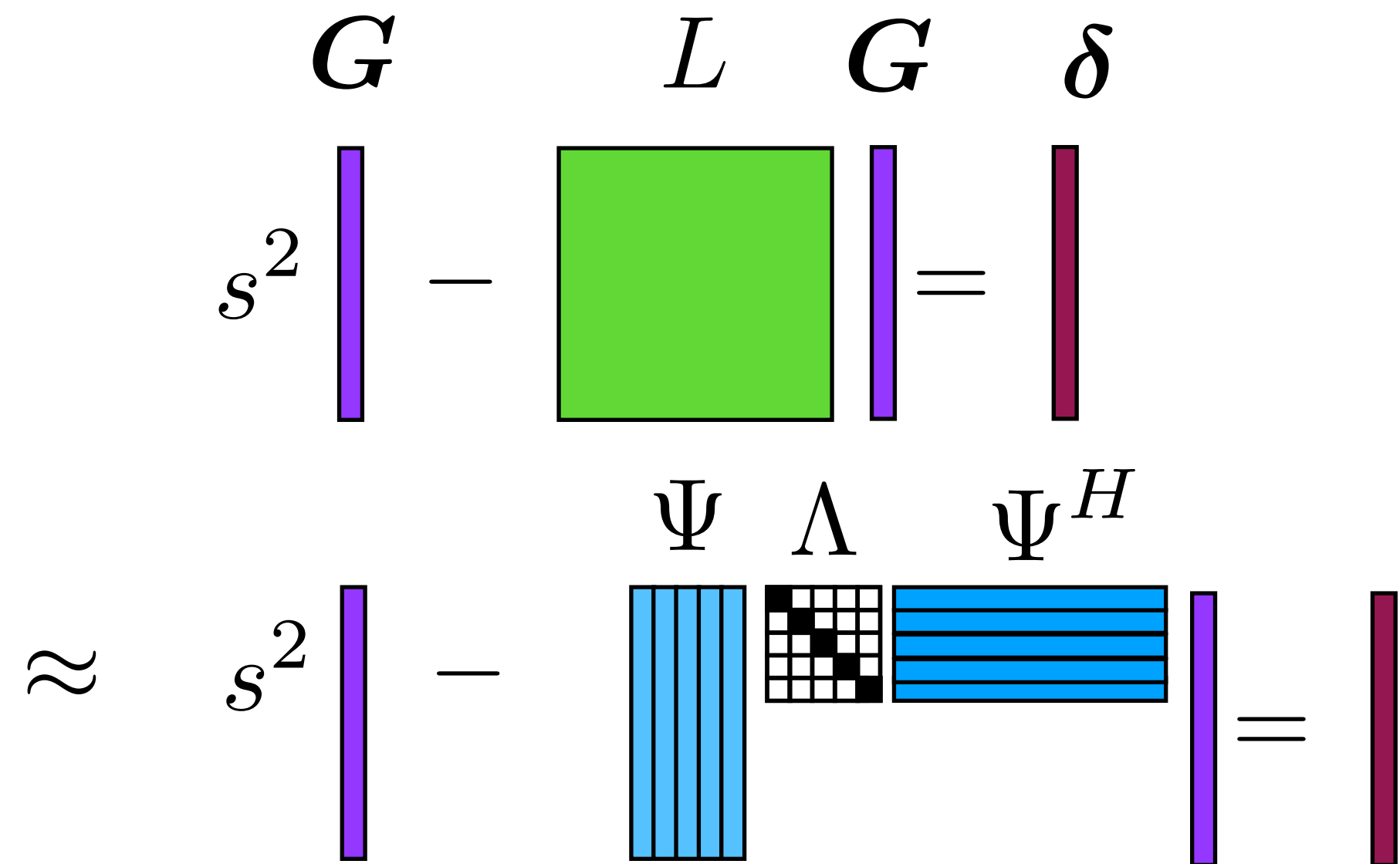
L can be approximated by



Note that each column ψ_k of Ψ is an eigenmode of L

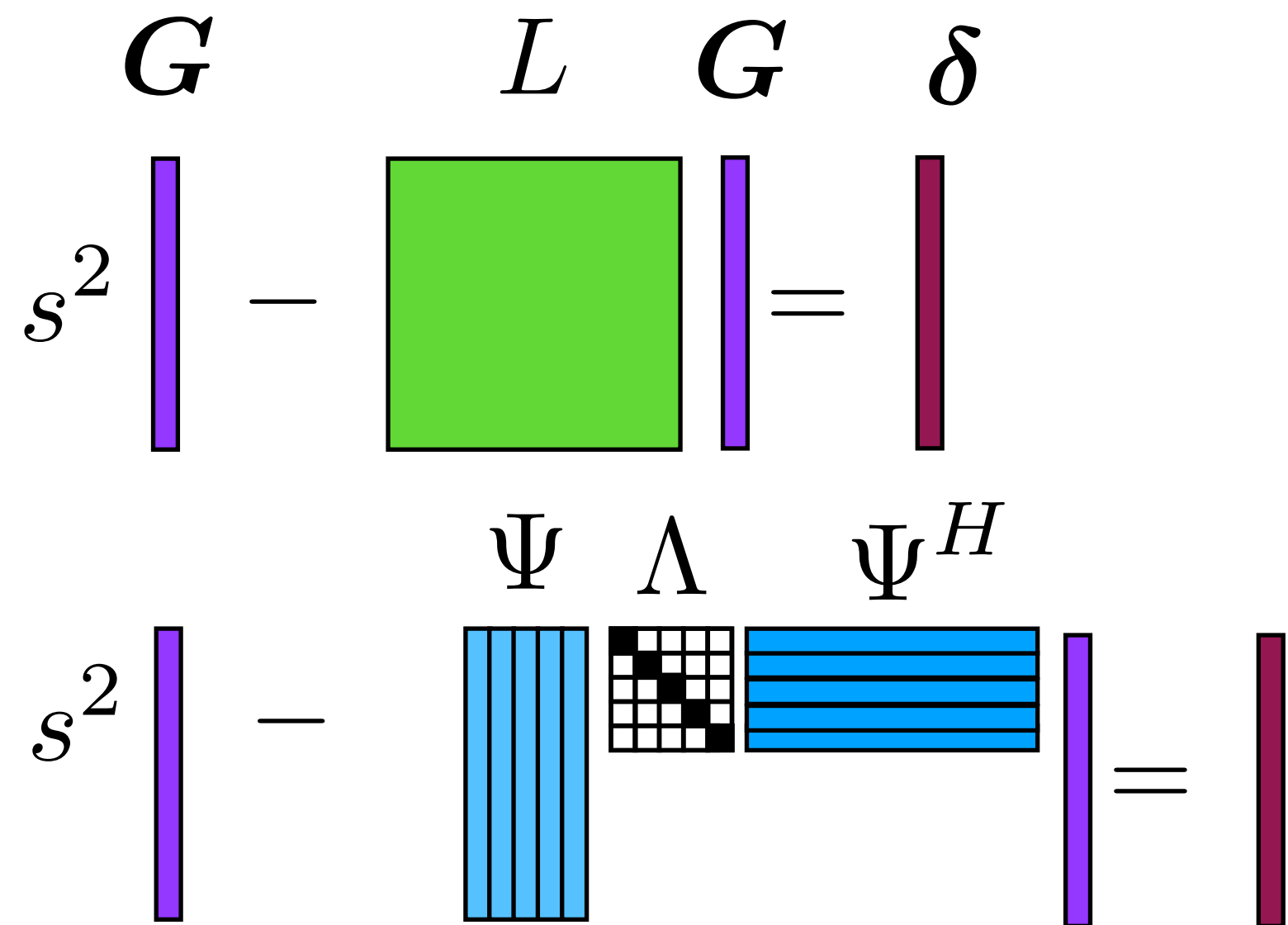
\downarrow
 passive acoustic mode of the system, where $\hat{q} = 0$

After discretization of $s^2 \hat{G} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{G}}{\partial x_i} \right)}_{\mathcal{L} \hat{G}} = \delta(x - x_0)$



We assume that G is a superposition of passive acoustic modes

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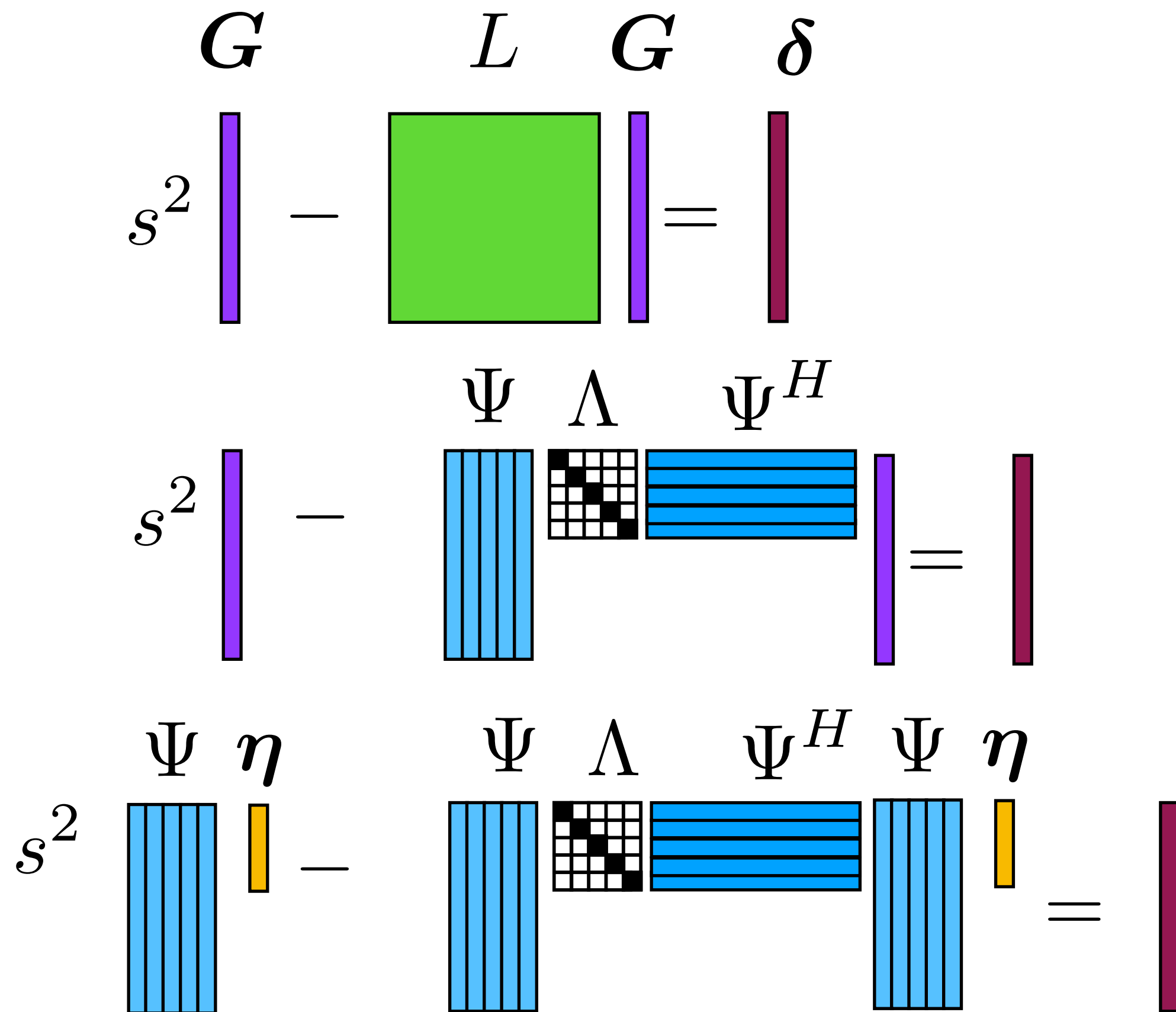


$$G = \sum_{k=1}^N \eta_k \psi_k = \Psi \eta$$

Diagram illustrating the modal expansion of G . A vertical purple bar G is equal to a matrix of vertical blue bars Ψ multiplied by a vertical yellow bar η .

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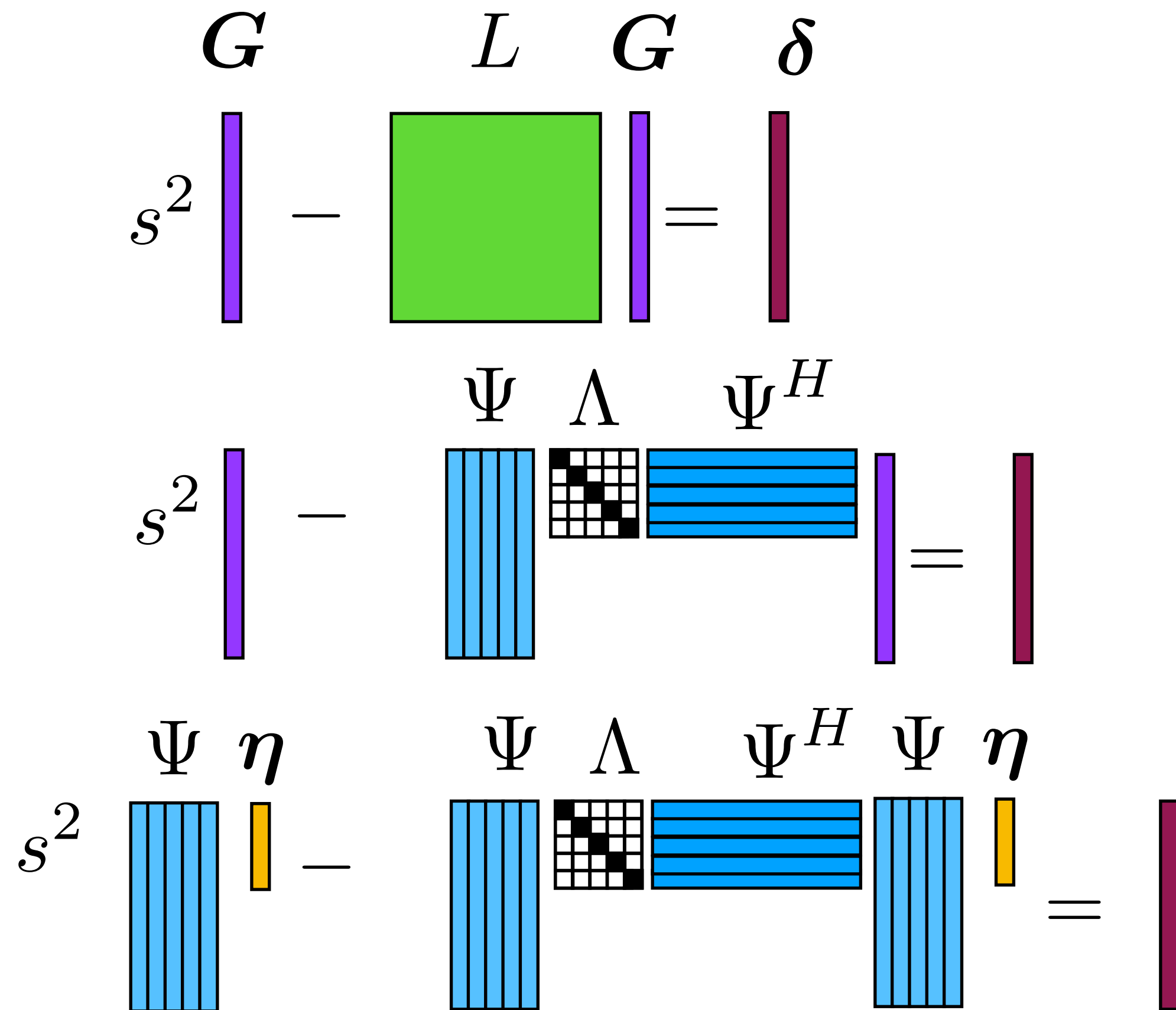


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$G = \Psi \eta$

We exploit now the orthogonality (orthonormality) of the eigenmodes

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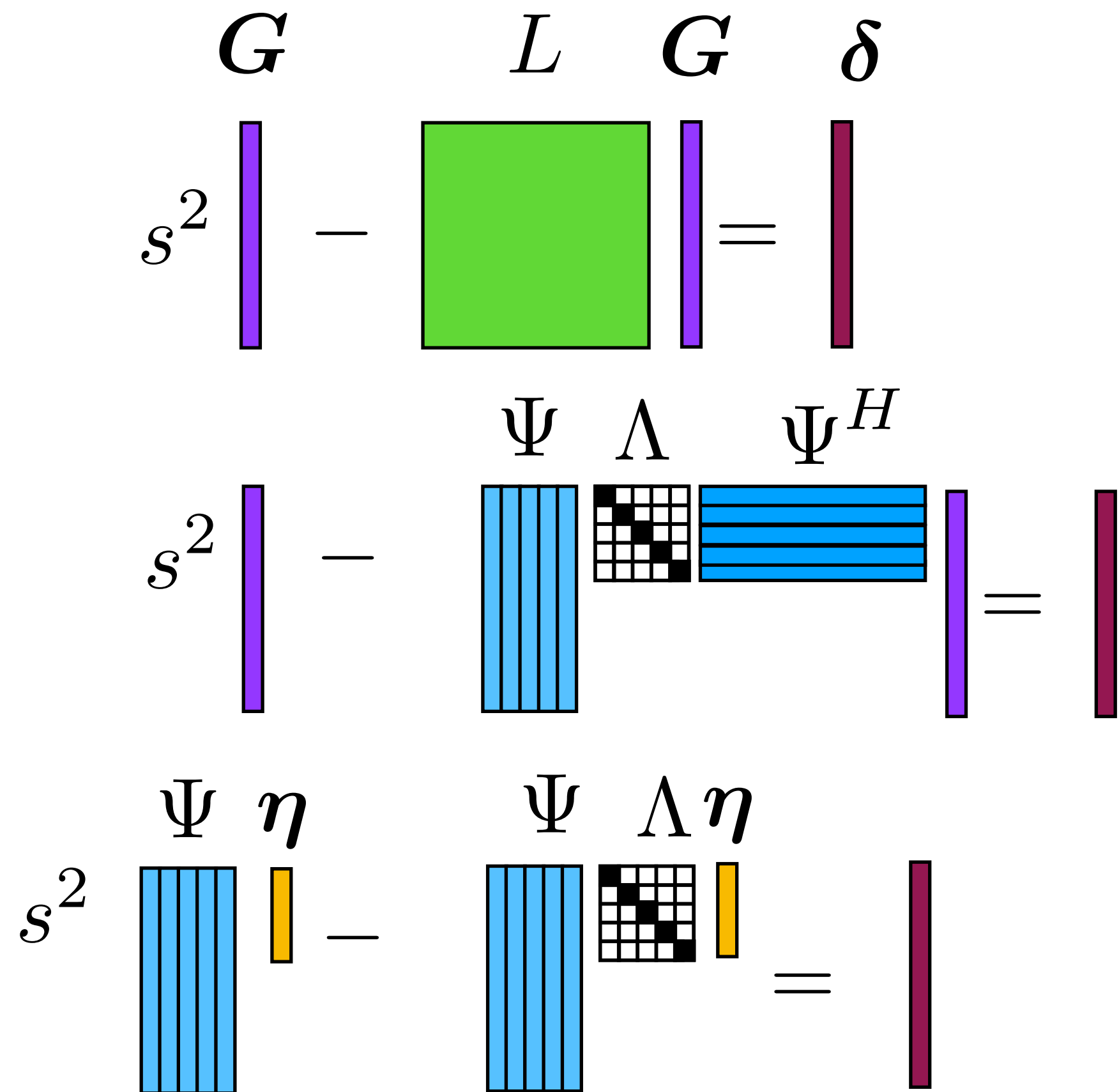


$$G = \sum_{k=1}^N \eta_k \psi_k = \Psi \eta$$

The diagram shows the modal expansion $G = \Psi \eta$. G is a purple vertical bar, Ψ is a blue vertical bar with vertical lines, and η is a yellow vertical bar.

We exploit now the orthogonality (orthonormality) of the eigenmodes

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$$\mathbf{G} = \sum_{k=1}^N \eta_k \boldsymbol{\psi}_k = \boldsymbol{\Psi} \boldsymbol{\eta}$$

$\mathbf{G} = \boldsymbol{\Psi} \boldsymbol{\eta}$

We exploit now the orthogonality (orthonormality) of the eigenmodes

$$s^2 \Psi \eta - \Psi \Lambda \eta = \delta$$

We exploit now the orthogonality (orthonormality) of the eigenmodes

$$s^2 \begin{matrix} \Psi & \eta \\ \text{[blue vertical bars]} & \text{[yellow vertical bar]} \end{matrix} - \begin{matrix} \Psi & \Lambda \eta \\ \text{[blue vertical bars]} & \begin{matrix} \text{[grid]} \\ \text{[yellow vertical bar]} \end{matrix} \end{matrix} = \begin{matrix} \delta \\ \text{[purple vertical bar]} \end{matrix}$$

$$s^2 \begin{matrix} \Psi^H & \Psi & \eta \\ \text{[blue horizontal bars]} & \text{[blue vertical bars]} & \text{[yellow vertical bar]} \end{matrix} - \begin{matrix} \Psi^H & \Psi & \Lambda \eta \\ \text{[blue horizontal bars]} & \text{[blue vertical bars]} & \begin{matrix} \text{[grid]} \\ \text{[yellow vertical bar]} \end{matrix} \end{matrix} = \begin{matrix} \Psi^H \\ \text{[blue horizontal bars]} & \text{[purple vertical bar]} \end{matrix}$$

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$$s^2 \begin{matrix} \eta \\ \text{yellow vertical bar} \end{matrix} - \begin{matrix} \Lambda \eta \\ \begin{matrix} \text{black and white grid} \\ \text{yellow vertical bar} \end{matrix} \end{matrix} = \begin{matrix} \Psi^H \\ \text{blue horizontal bars} \end{matrix} \begin{matrix} \\ \text{purple vertical bar} \end{matrix}$$

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$$s^2 \begin{matrix} \Psi & \eta \\ \text{blue vertical bars} & \text{yellow vertical bar} \end{matrix} - \begin{matrix} \Psi & \Lambda \eta \\ \text{blue vertical bars} & \text{black and white checkerboard grid} \end{matrix} = \begin{matrix} \delta \\ \text{purple vertical bar} \end{matrix}$$

$$s^2 \begin{matrix} \eta \\ \text{yellow vertical bar} \end{matrix} - \begin{matrix} \Lambda \eta \\ \text{black and white checkerboard grid} \end{matrix} = \begin{matrix} \Psi^H \\ \text{blue horizontal bars} \end{matrix} \Rightarrow \begin{matrix} \eta \\ \text{yellow vertical bar} \end{matrix} = \left(\begin{matrix} I & \\ \text{red and white checkerboard grid} & s^2 \end{matrix} - \begin{matrix} \Lambda \\ \text{black and white checkerboard grid} \end{matrix} \right)^{-1} \begin{matrix} \Psi^H \\ \text{blue horizontal bars} \end{matrix}$$

By doing some linear algebra we have solved the Equation!!

$$s^2 \begin{matrix} \Psi & \eta \\ \text{blue vertical bars} & \text{yellow vertical bar} \end{matrix} - \begin{matrix} \Psi & \Lambda \eta \\ \text{blue vertical bars} & \text{black and white checkerboard grid} \end{matrix} = \begin{matrix} \delta \\ \text{purple vertical bar} \end{matrix}$$

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recall that

$$G = \begin{matrix} \Psi & \eta \\ \text{blue vertical bars} & \text{yellow vertical bar} \end{matrix} \Rightarrow G = \left(\begin{matrix} I & \\ \text{red and white checkerboard grid} & s^2 \end{matrix} - \begin{matrix} \Lambda \\ \text{black and white checkerboard grid} \end{matrix} \right)^{-1} \begin{matrix} \Psi & \Psi^H \\ \text{blue vertical bars} & \text{blue horizontal bars} \end{matrix}$$

The solution of the Helmholtz equation reads

Helmholtz Equation

$$s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = \hat{h}$$

Boundary Conditions

$$\mathbf{n} \cdot \frac{\partial \hat{p}}{\partial x_i} = -\hat{f}$$

Solution

$$\hat{p} = \int_V G h dV + \int_{\partial V} G f dS$$

$$\begin{array}{c} \mathbf{p} \\ | \\ \mathbf{I} \end{array} = \left(\begin{array}{c} I \\ \text{grid} \end{array} s^2 - \begin{array}{c} \Lambda \\ \text{grid} \end{array} \right)^{-1} \begin{array}{c} \Psi \\ \text{grid} \end{array} \begin{array}{c} \Psi^H \\ \text{grid} \end{array} \begin{array}{c} \mathbf{h} \\ | \\ \text{orange} \end{array} + \underbrace{\left[\begin{array}{c} I \\ \text{grid} \end{array} s^2 - \begin{array}{c} \Lambda \\ \text{grid} \end{array} \right)^{-1} \begin{array}{c} \Psi \\ \text{grid} \end{array} \begin{array}{c} \Psi^H \\ \text{grid} \end{array} \begin{array}{c} \mathbf{f} \\ | \\ \text{green} \end{array} \right]}_{\text{evaluated at BC}}$$

what about the state space approach?

Outline

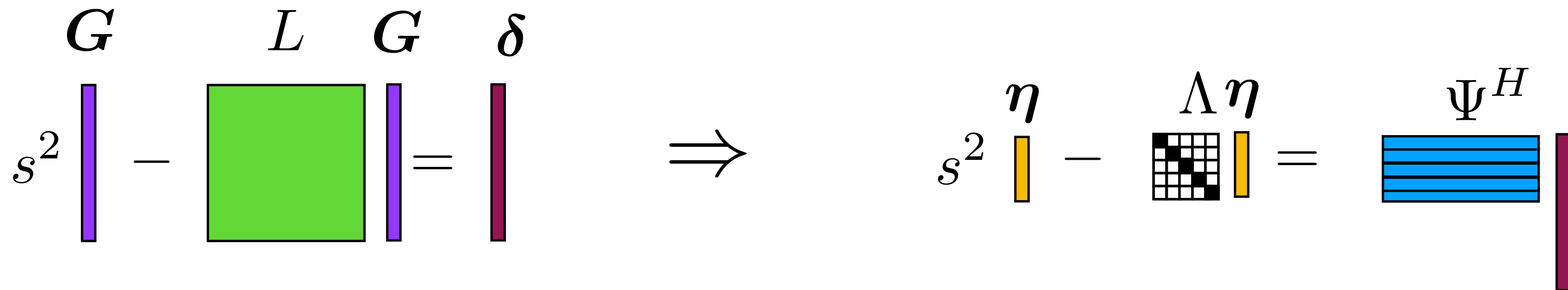
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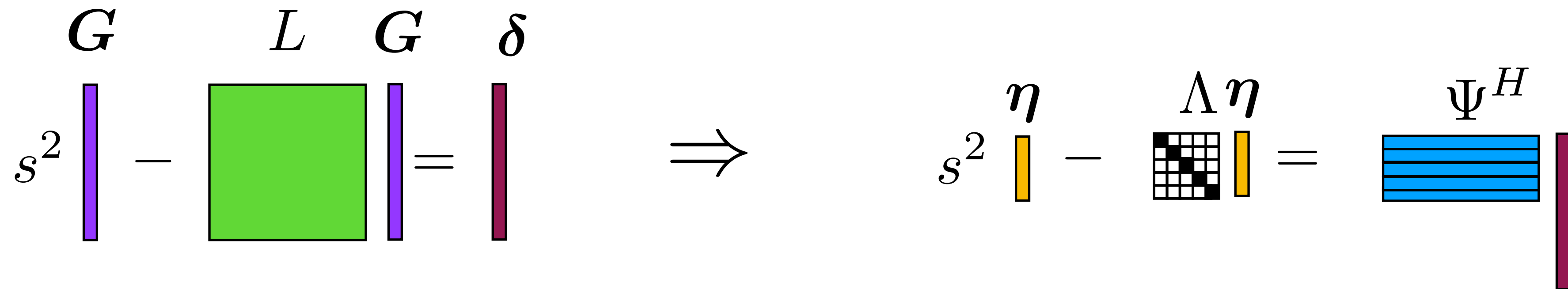
We can recover a differential equation for η

After discretization of $s^2 \hat{G} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{G}}{\partial x_i} \right)}_{\mathcal{L} \hat{G}} = \delta(x - x_0)$



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The dynamics of η is governed by

$$\ddot{\eta} - \Lambda \eta = \Psi^H \delta$$

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$$s^2 \begin{matrix} G \\ \text{purple bar} \end{matrix} - \begin{matrix} L \\ \text{green square} \end{matrix} \begin{matrix} G \\ \text{purple bar} \end{matrix} = \begin{matrix} \delta \\ \text{red bar} \end{matrix} \quad \Rightarrow \quad s^2 \begin{matrix} \eta \\ \text{yellow bar} \end{matrix} - \begin{matrix} \Lambda \eta \\ \text{black grid} \end{matrix} = \begin{matrix} \Psi^H \\ \text{blue horizontal bars} \end{matrix} \begin{matrix} \delta \\ \text{red bar} \end{matrix}$$

The dynamics of η is governed by

$$\ddot{\eta} - \Lambda \eta = \Psi^H \delta \quad \Rightarrow \quad \begin{matrix} \dot{\eta} = v \\ \dot{v} = \Lambda \eta + \Psi^H \delta \end{matrix}$$

↑
state vector

We can recover a differential equation for η

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These Equations are the first step towards the state space representation

Does makes to consider only one mode in the expansion?

Yes!

Outline

† Solving the Helmholtz Equation by modal expansion

† And the state space?

† About a one mode expansion

Let us assume homogeneous Neumann BC (no flux)

Solution of $s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = s(\gamma - 1) \hat{q}$

$$s^2 \begin{array}{|c} p \\ \hline \end{array} - \begin{array}{|c|} \hline \square \\ \hline \end{array} \begin{array}{|c} p \\ \hline \end{array} = s(\gamma - 1) \begin{array}{|c} q \\ \hline \end{array} \Rightarrow s^2 \begin{array}{|c} \eta \\ \hline \end{array} - \begin{array}{|c|} \hline \text{grid} \\ \hline \end{array} \begin{array}{|c} \eta \\ \hline \end{array} = \begin{array}{|c|} \hline \Psi^H \\ \hline \end{array} \begin{array}{|c} q \\ \hline \end{array}$$

Consider only one mode in the approximation

$$s^2 \begin{array}{|c} \eta_k \\ \hline \end{array} - \begin{array}{|c|} \hline \lambda_k^2 \\ \hline \end{array} \begin{array}{|c} \eta_k \\ \hline \end{array} = s(\gamma - 1) \begin{array}{|c|} \hline \psi_k^H \\ \hline \end{array} \begin{array}{|c} q \\ \hline \end{array}$$

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$$s^2 \eta_k - \lambda_k^2 \eta_k = s(\gamma - 1) \psi_k^H q$$

Let us assume homogeneous Neumann BC (no flux)

We have gone from here

$$s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = s(\gamma - 1) \hat{q}$$

to here

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to here

$$s^2 \eta_k - \lambda_k^2 \eta_k = s(\gamma - 1) \psi_k^H \mathbf{q}$$

Consider now two models for the flame response

model 1

$$\mathbf{q} = I_f I_{\text{ref}}^\top \mathbf{p} n_p e^{-s\tau_p} \approx \eta_k I_f I_{\text{ref}}^\top \psi_k n_p e^{-s\tau_p}$$

Let us assume homogeneous Neumann BC (no flux)

We have gone from here $s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = s(\gamma - 1) \hat{q}$

to here $s^2 \eta_k - \lambda_k^2 \eta_k = s(\gamma - 1) \psi_k^H \mathbf{q}$

Consider now two models for the flame response

model 1

$$\mathbf{q} = I_f I_{\text{ref}}^\top \mathbf{p} n_p e^{-s\tau_p} \approx \eta_k I_f I_{\text{ref}}^\top \psi_k n_p e^{-s\tau_p} \implies s^2 - \lambda_k^2 = s(\gamma - 1) \kappa n_p e^{-s\tau_p}$$

Let us assume homogeneous Neumann BC (no flux)

We have gone from here $s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i} \left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i} \right)}_{\mathcal{L} \hat{p}} = s(\gamma - 1) \hat{q}$

to here $s^2 \eta_k - \lambda_k^2 \eta_k = s(\gamma - 1) \psi_k^H \mathbf{q}$

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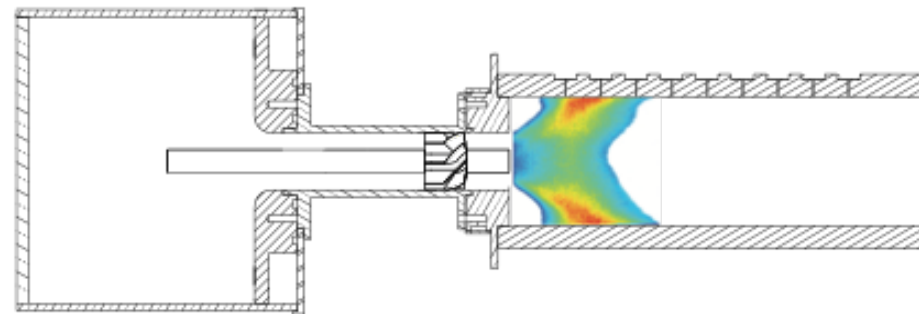
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For the case investigated, the agreement of the two approaches is remarkable

system under study



Helmholtz Equation

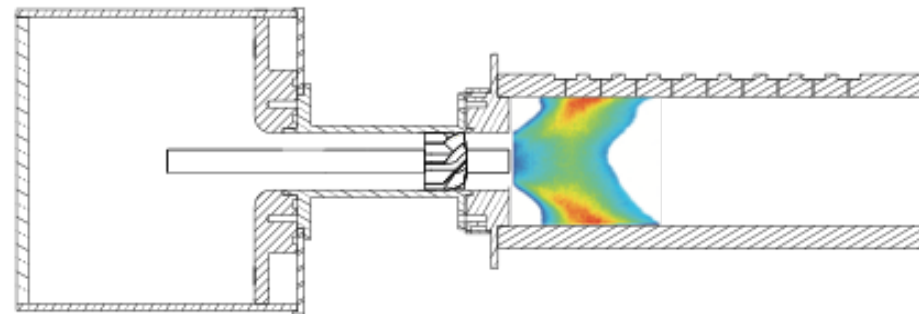
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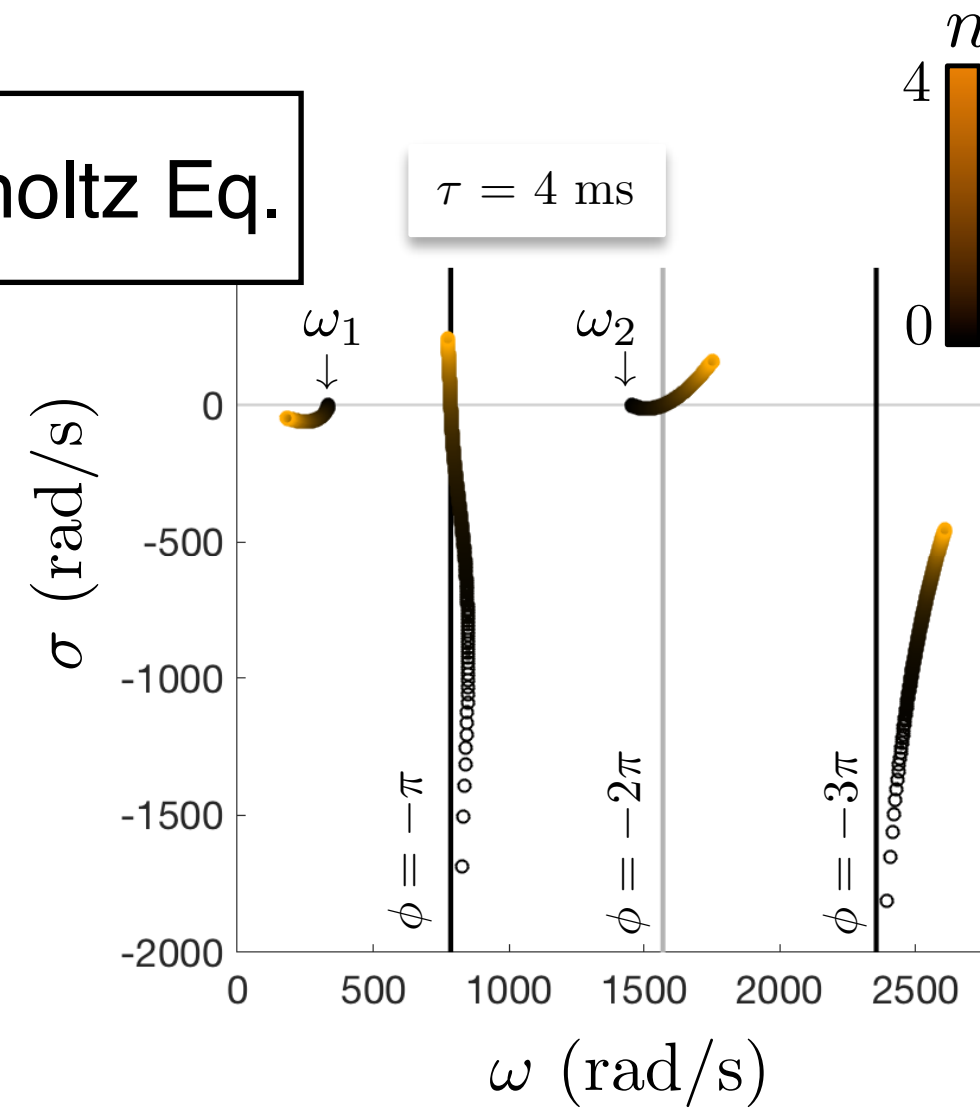
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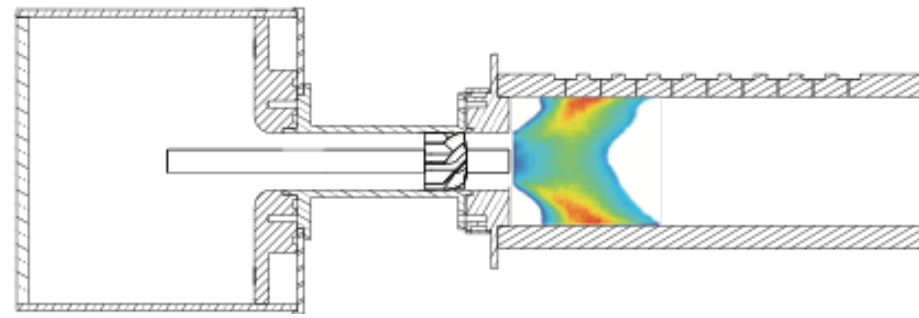
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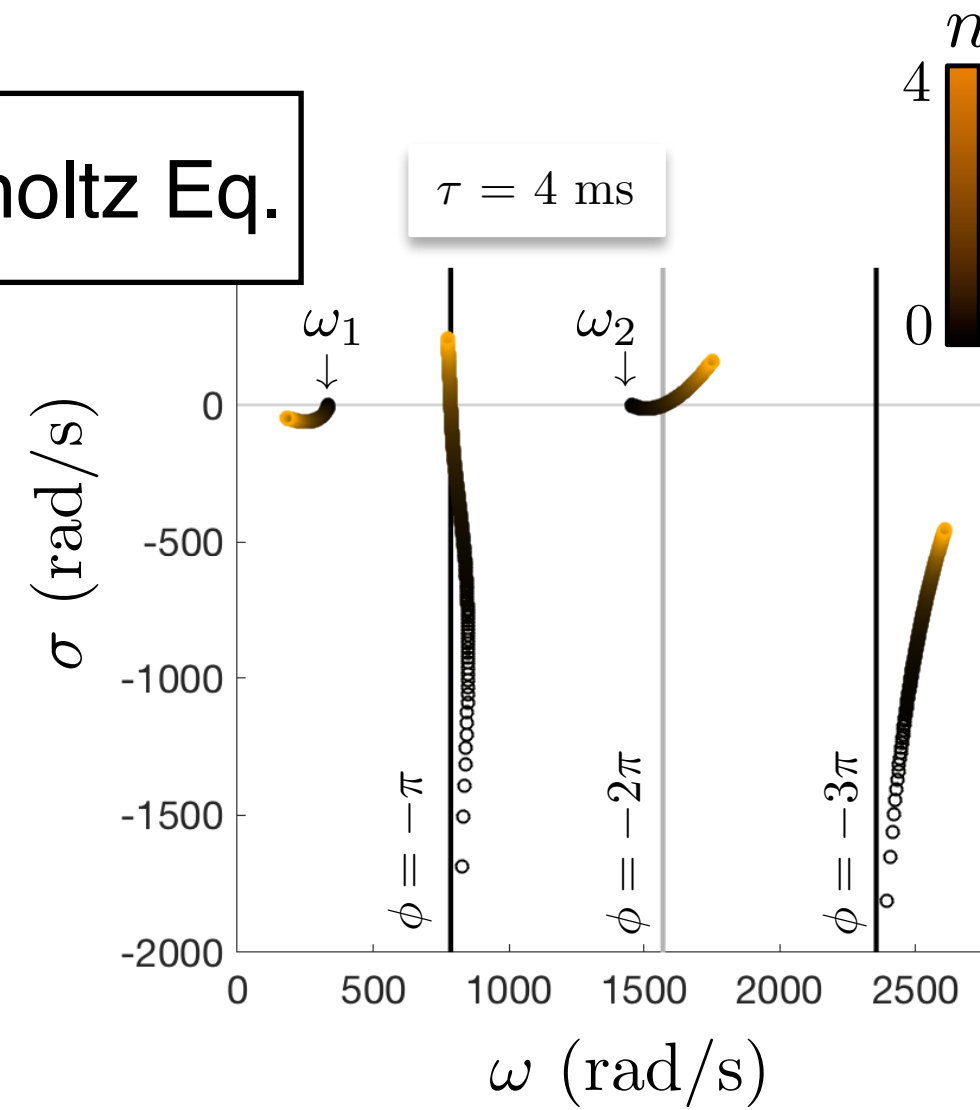
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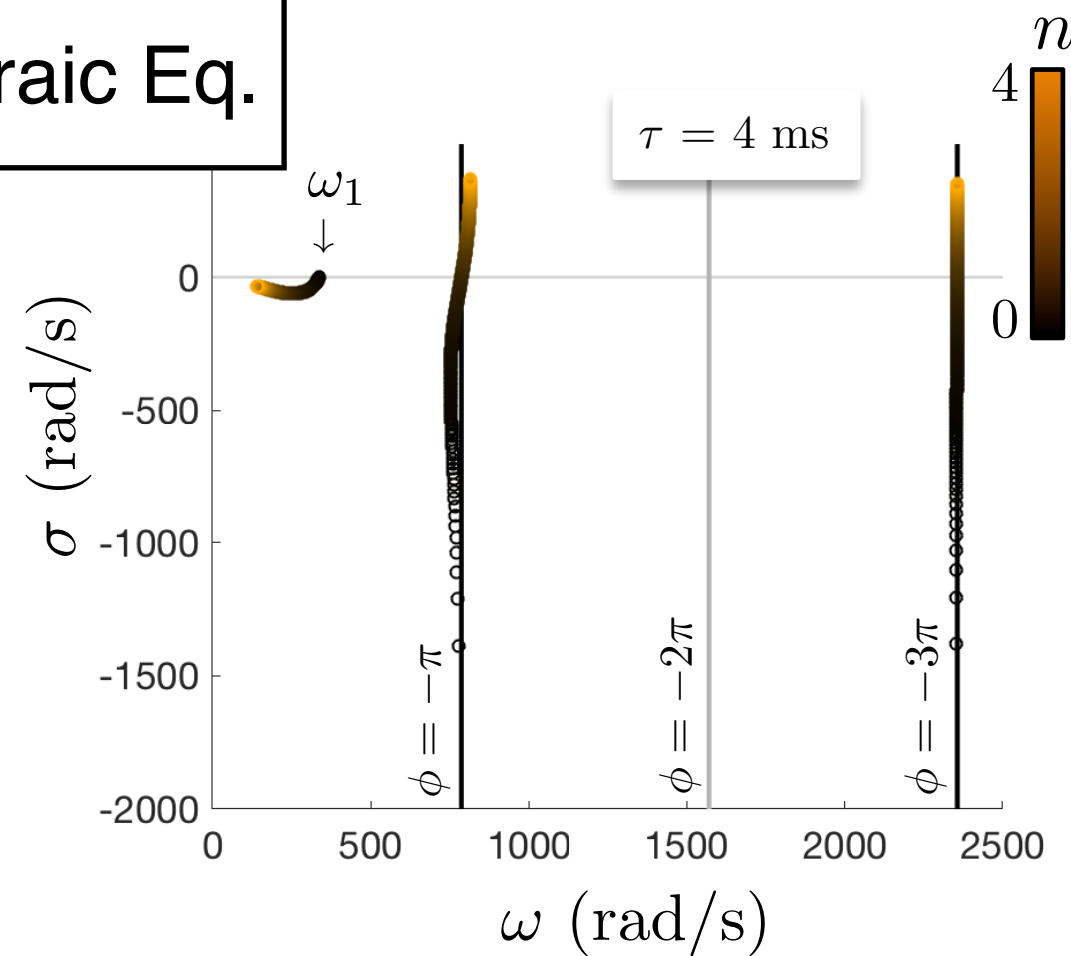
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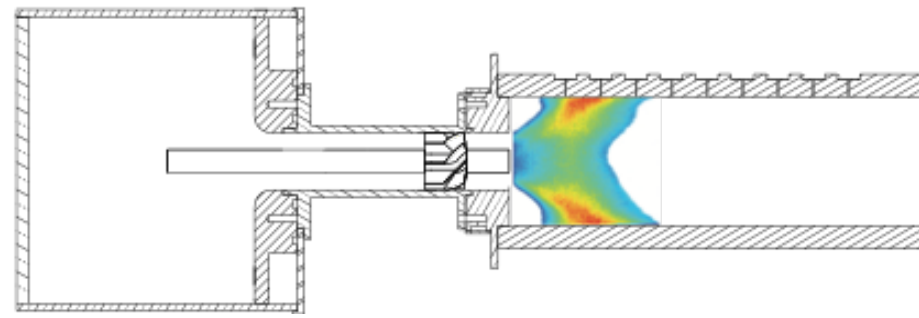
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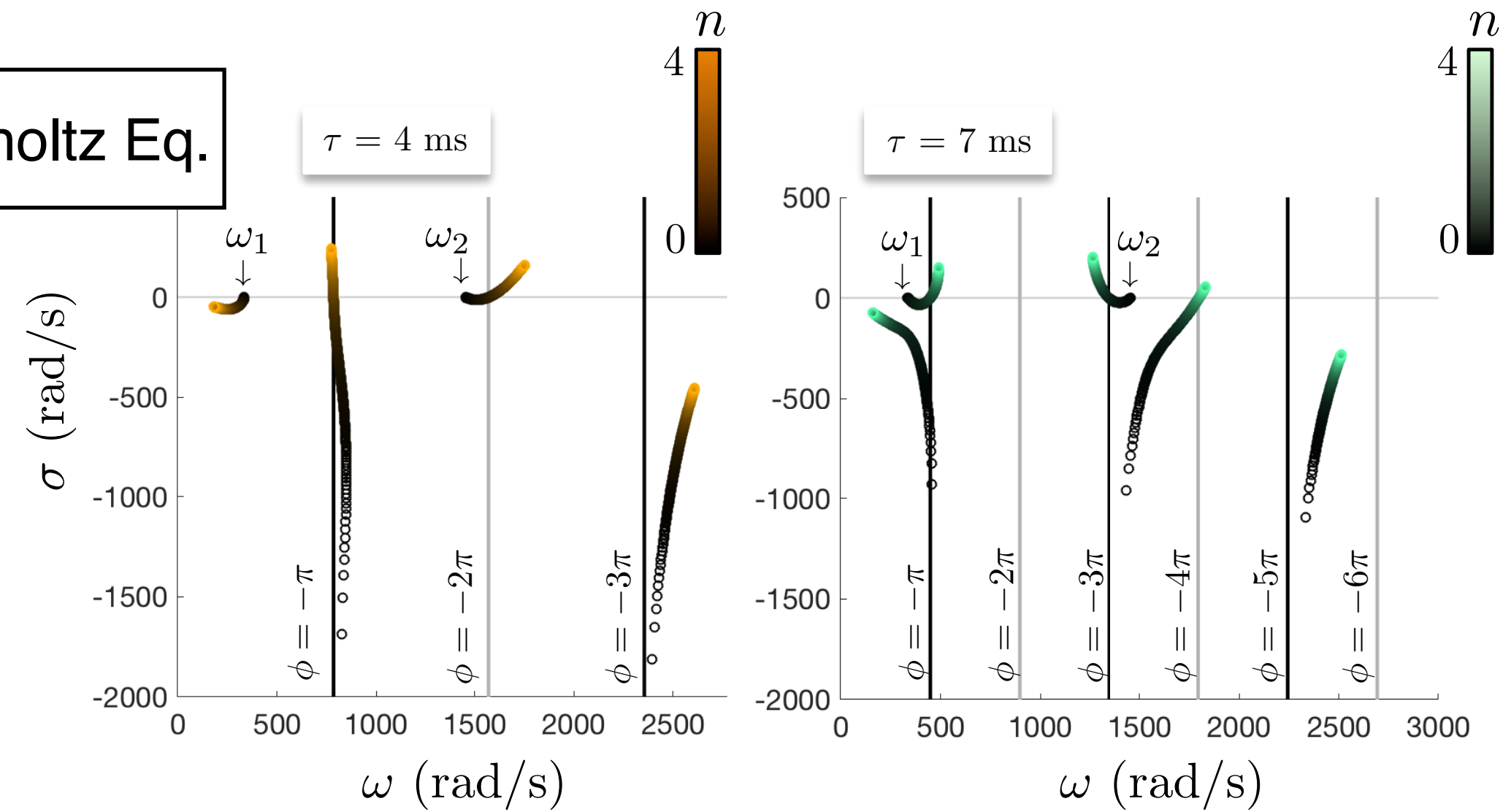


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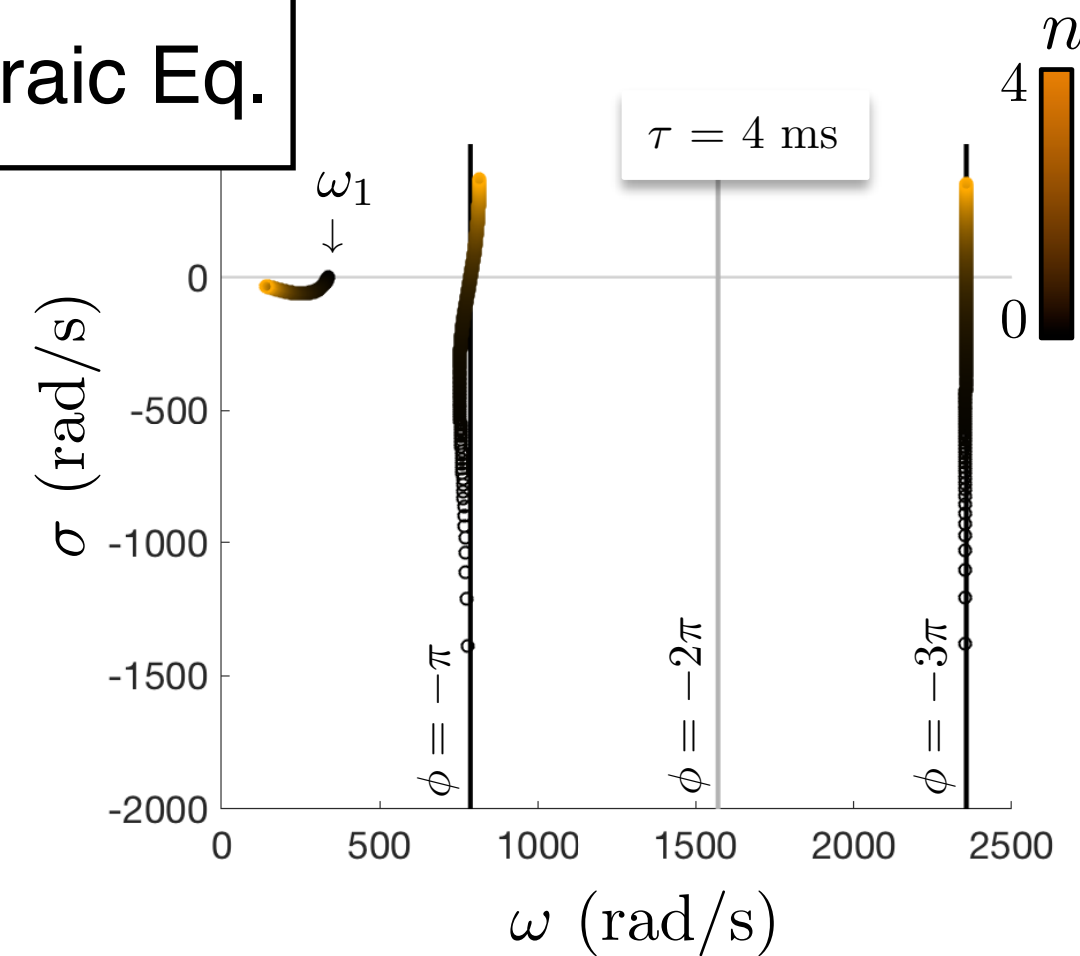
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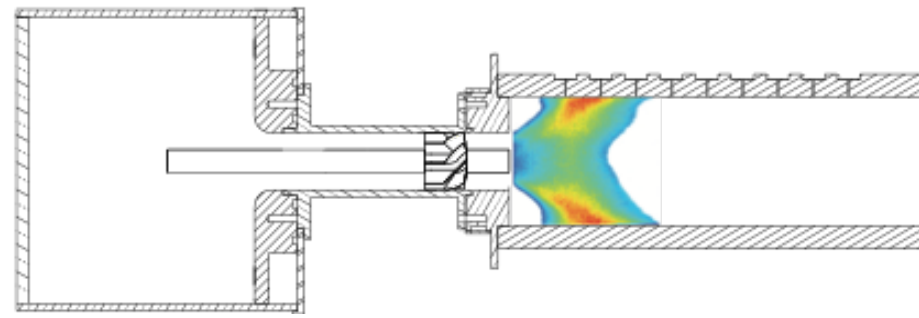
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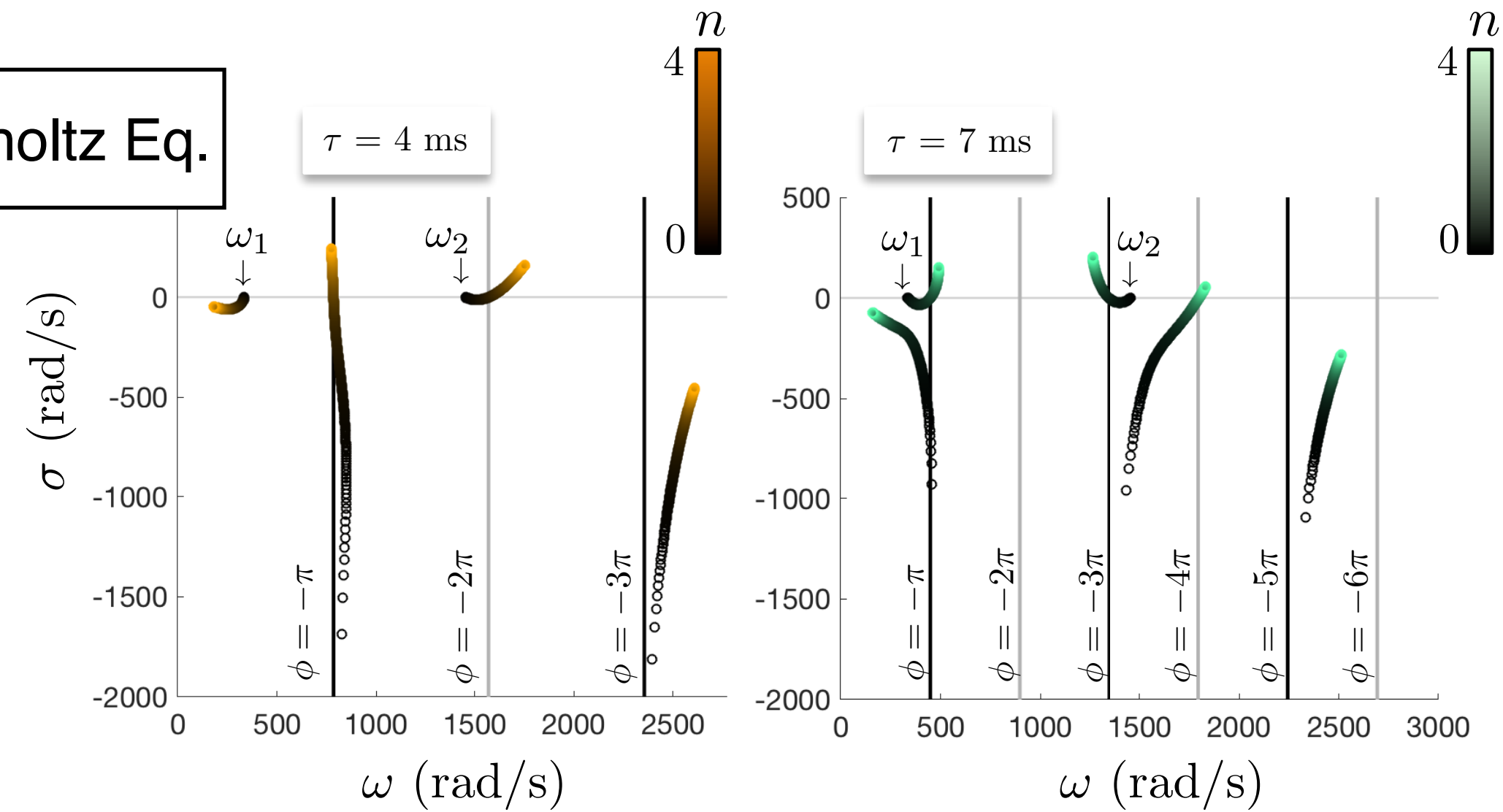


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