The Galerkin approach to ‘reduce’ models in thermoacoustics

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Even if LNSE or the Helmholtz equation are expressed in a state-space framework, they remain computationally expensive.

Linearized Navier Stokes Equations

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho} u_j' + \rho' \bar{u}_j\right) = 0$$

$$\frac{\partial}{\partial t}\left(\bar{\rho} u_i' + \rho' \bar{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho} \bar{u}_i u_j' + \bar{\rho} u_i' \bar{u}_j + \rho' \bar{u}_i \bar{u}_j\right) = -\frac{\partial p'}{\partial x_i} + \frac{\partial \tau_{ij}'}{\partial x_j}$$

$$\bar{T}\left[\frac{\partial}{\partial t}\left(\bar{\rho} s' + \rho' \bar{s}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho} \bar{u}_j s' + \bar{\rho} u_j' \bar{s} + \rho' \bar{u}_j \bar{s}\right)\right] + T' \frac{\partial}{\partial x_j}\left(\bar{\rho} \bar{u}_j \bar{s}\right) = \dot{q}'$$

Helmholtz Equation

$$s^2 \hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}$$

How to model systems that are complex in frameworks that are easy for computation?

Let us take a look at reduced order models

Outline

- Solving the Helmholtz Equation by modal expansion
- And the state space?
- About a one mode expansion

The Helmholtz Equation is treated as the reference model to approximate

$$s^2\hat{p} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right)}_{\mathcal{L}\hat{p}} = \overbrace{s(\gamma - 1)\hat{\dot{q}}}^{\hat{h}}$$

Requires a flame response

Helmholtz Equation

$$s^2\hat{p} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right)}_{\mathcal{L}\hat{p}} = \hat{h}$$

Boundary Conditions

$$\mathbf{n} \cdot \frac{\partial \hat{p}}{\partial x_i} = -\hat{f}$$

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An auxiliary problem is required for the approximation

Helmholtz Equation

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Auxiliary Problem

$$s^2\hat{G} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2\frac{\partial \hat{G}}{\partial x_i}\right)}_{\mathcal{L}\hat{G}} = \delta(x - x_0)$$

$$\mathbf{n}\cdot\frac{\partial \hat{G}}{\partial x_i} = 0$$

The solution is given in terms of a Green's function *G*

Helmholtz Equation

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$$\mathbf{n}\cdot\frac{\partial \hat{G}}{\partial x_i} = 0$$

Solution

$$\hat{p} = \int_V G h \mathrm{d}V + \int_{\partial V} G f \mathrm{d}S$$

Let us solve the Auxiliary problem !

Helmholtz Equation

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In the Helmholtz Equation, the matrix L gathers all the information of the system

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operator matrix vector

$$\mathcal{L} \longrightarrow L$$

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operator		matrix	vector
$\mathcal{L}$	$\longrightarrow$	L	
$\hat{G}$	$\longrightarrow$ discretization		$\boldsymbol{G}$
$\delta(x - x_0)$	$\longrightarrow$		$\boldsymbol{\delta}$

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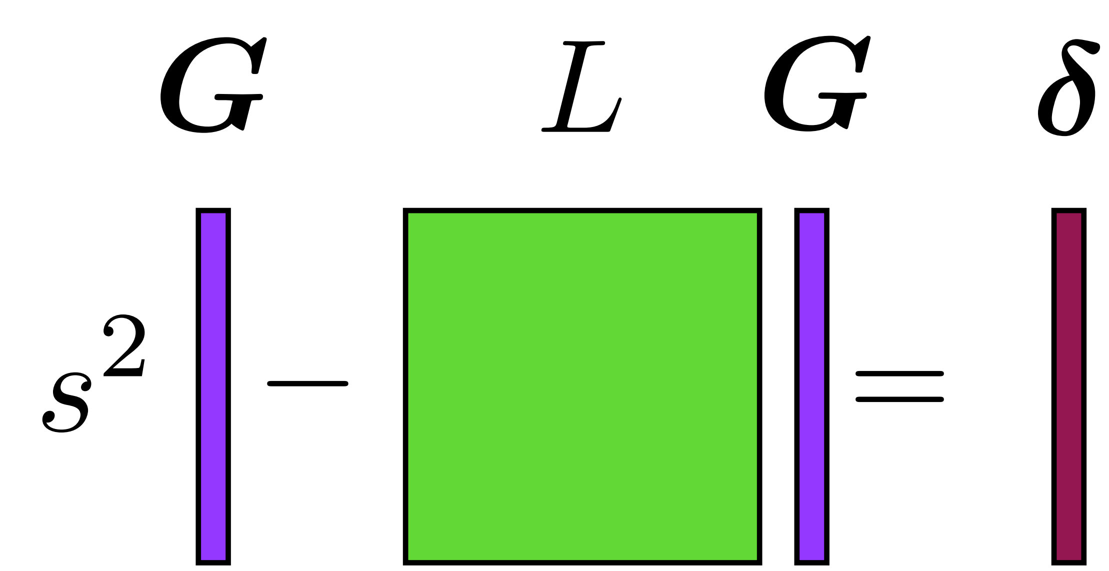

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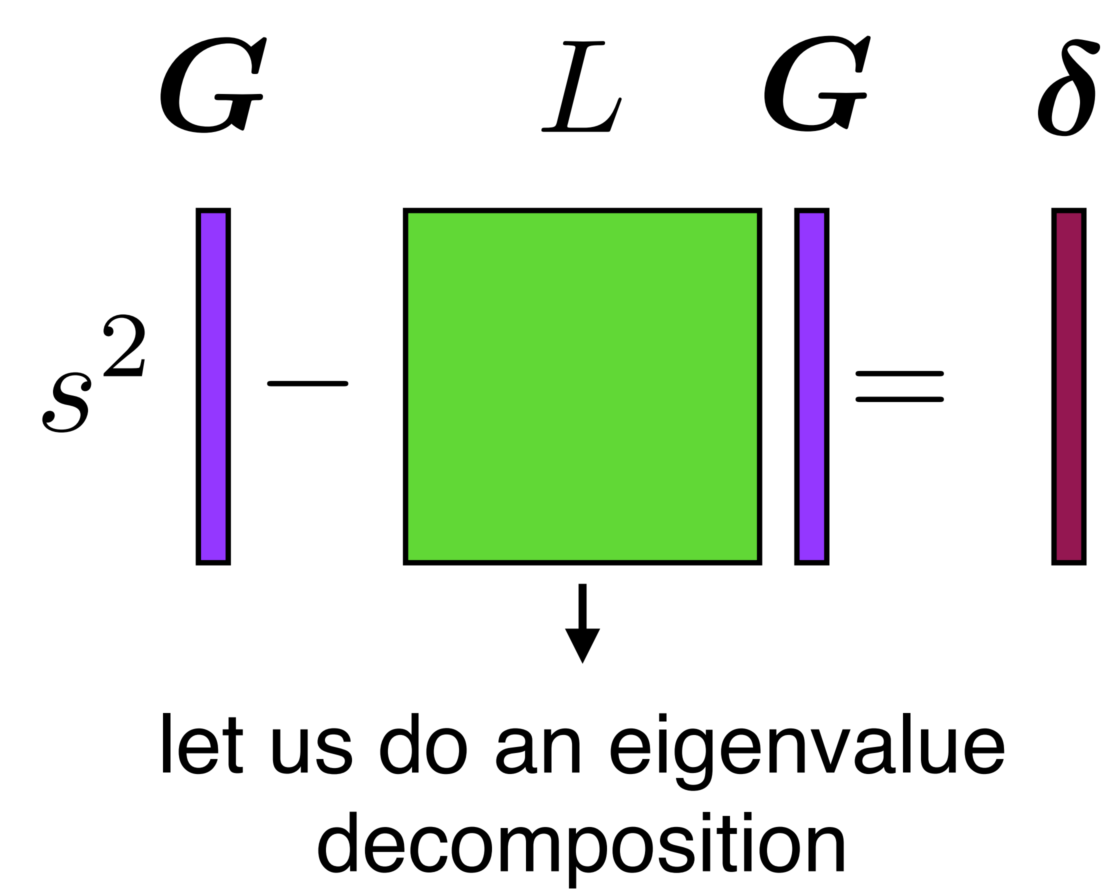
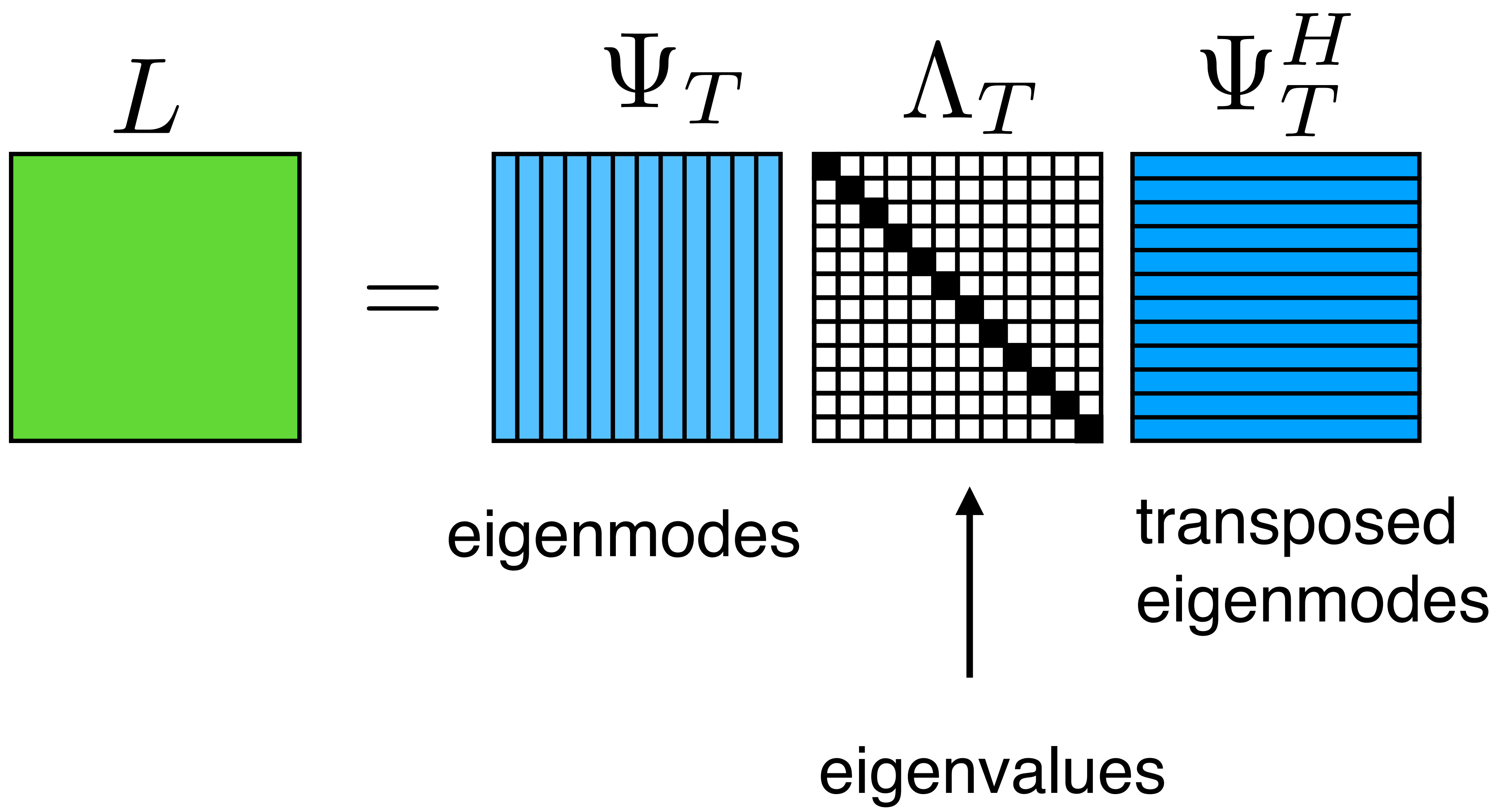
$\Rightarrow$



The auxiliary problem can be defined with ideal BC -> Self-adjoint Operator



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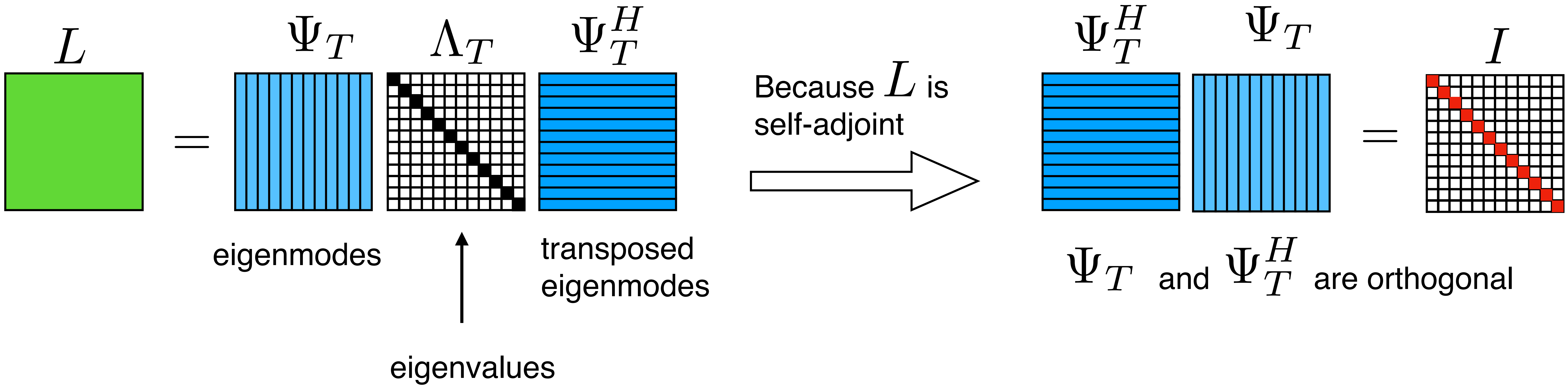

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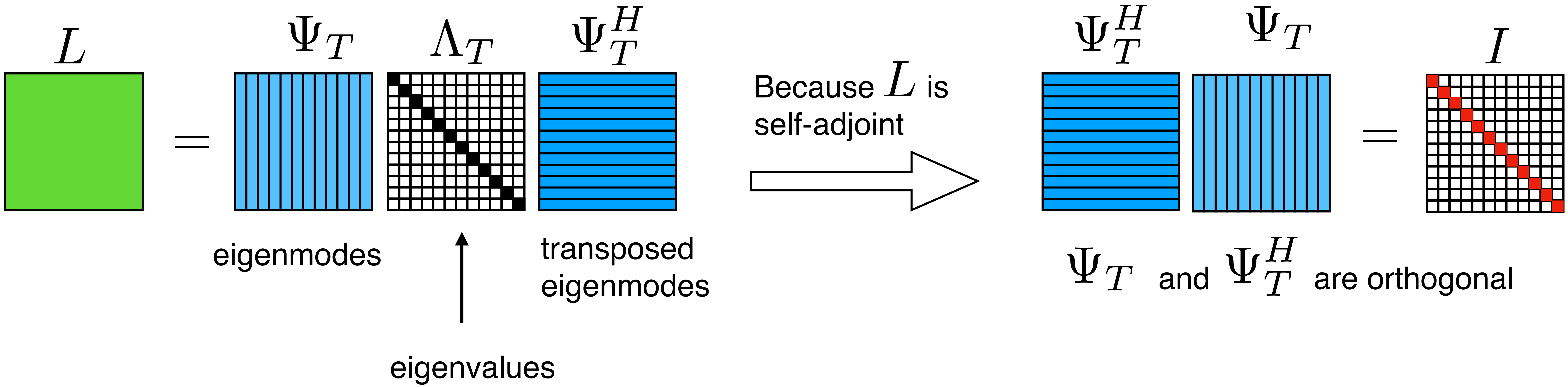
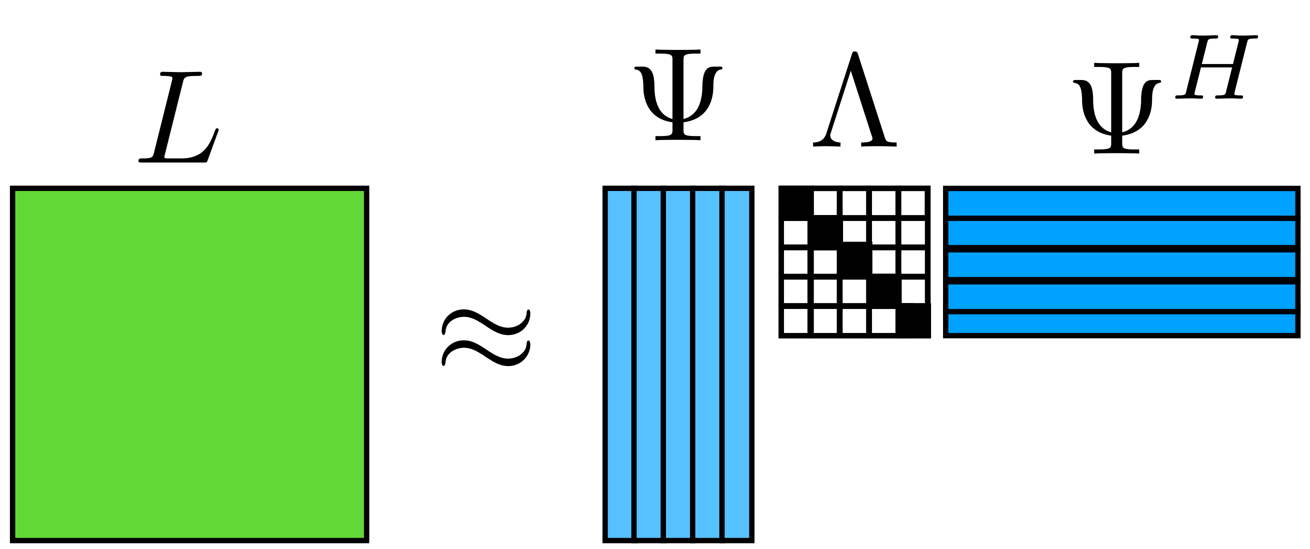


L can be approximated by



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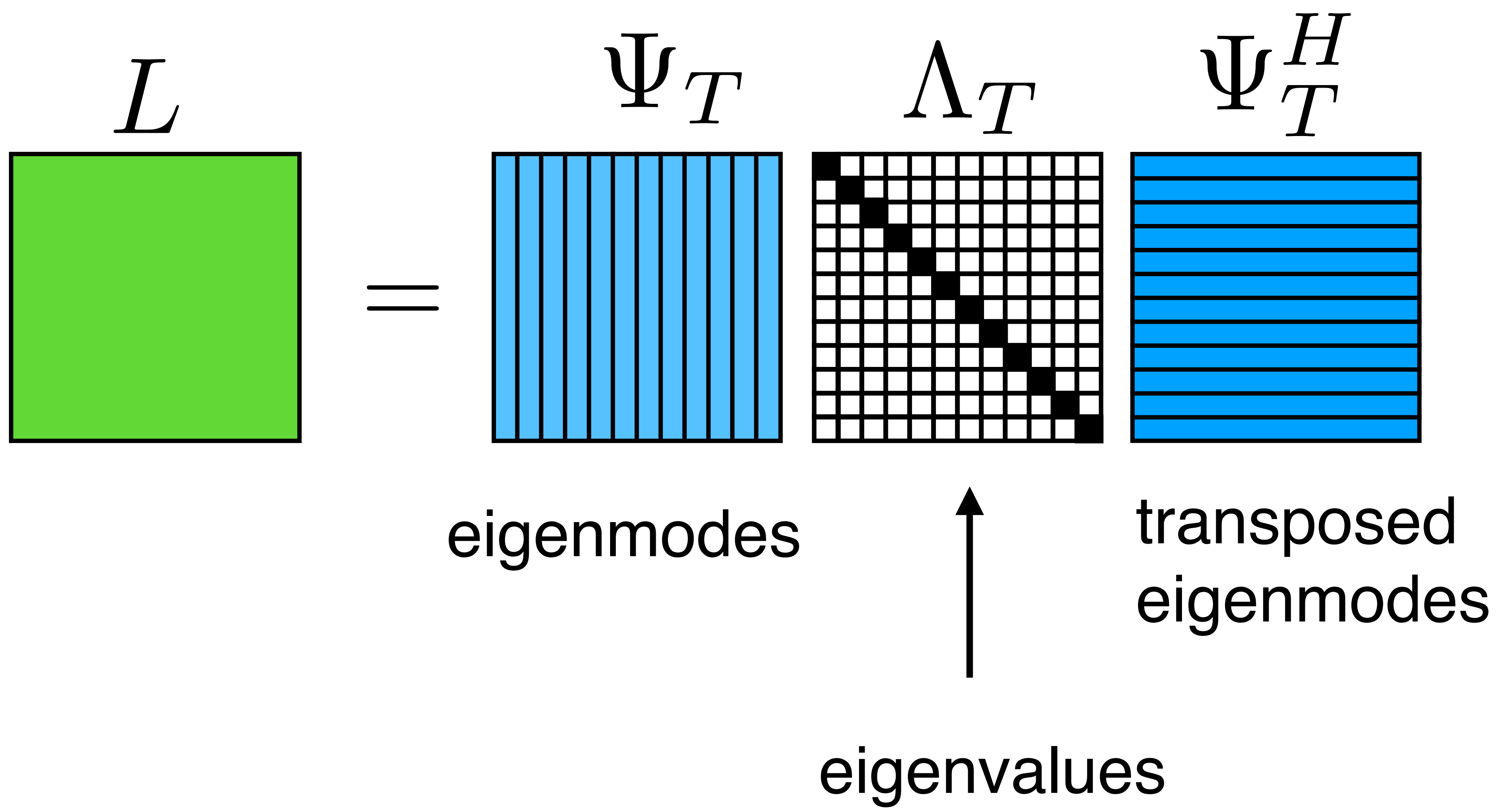
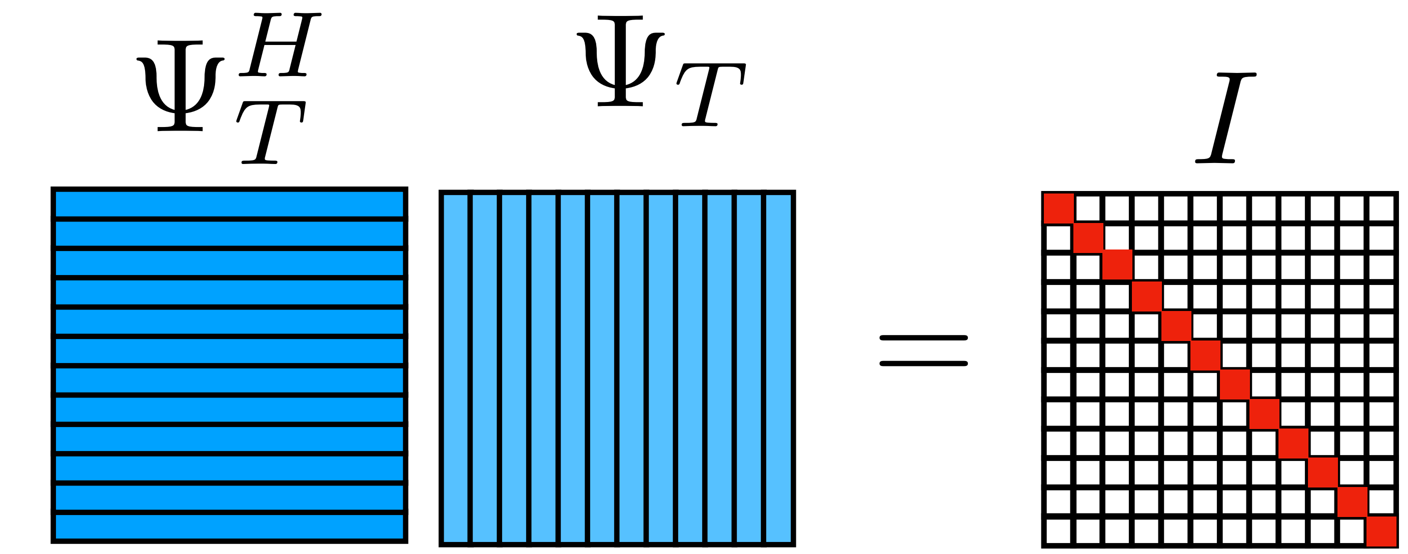
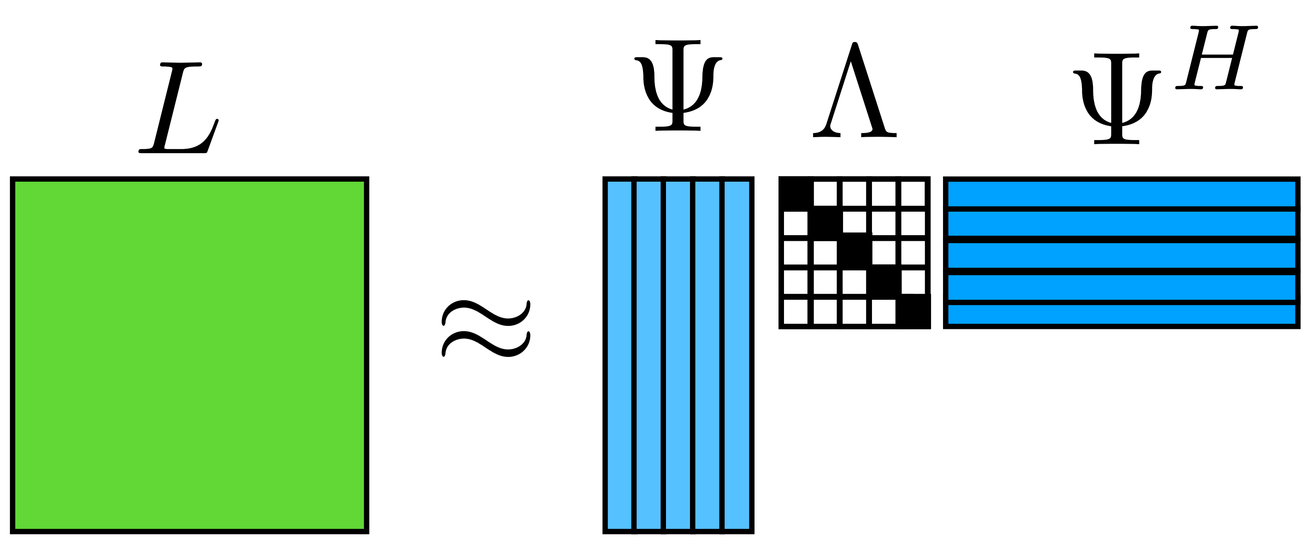


Because L is self-adjoint



Ψ_T and Ψ_T^H are orthogonal

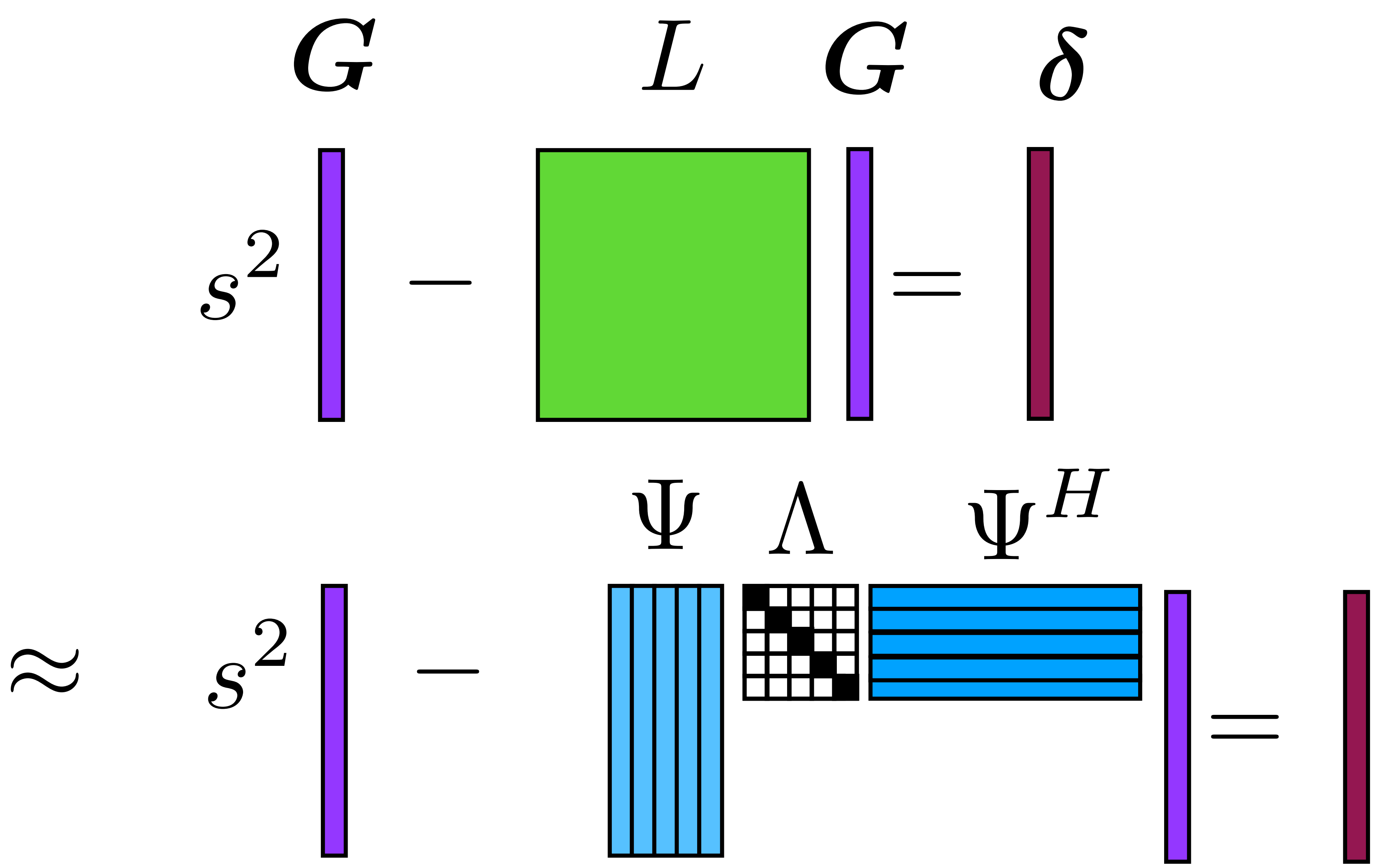
L can be approximated by



Note that each column ψ_k of Ψ is an eigenmode of L

passive acoustic mode of the system, where $\dot{\hat{q}} = 0$

After discretization of $$s^2\hat{G} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{G}}{\partial x_i}\right)}_{\mathcal{L}\hat{G}} = \delta(x - x_0)$$



We assume that G is a superposition of passive acoustic modes

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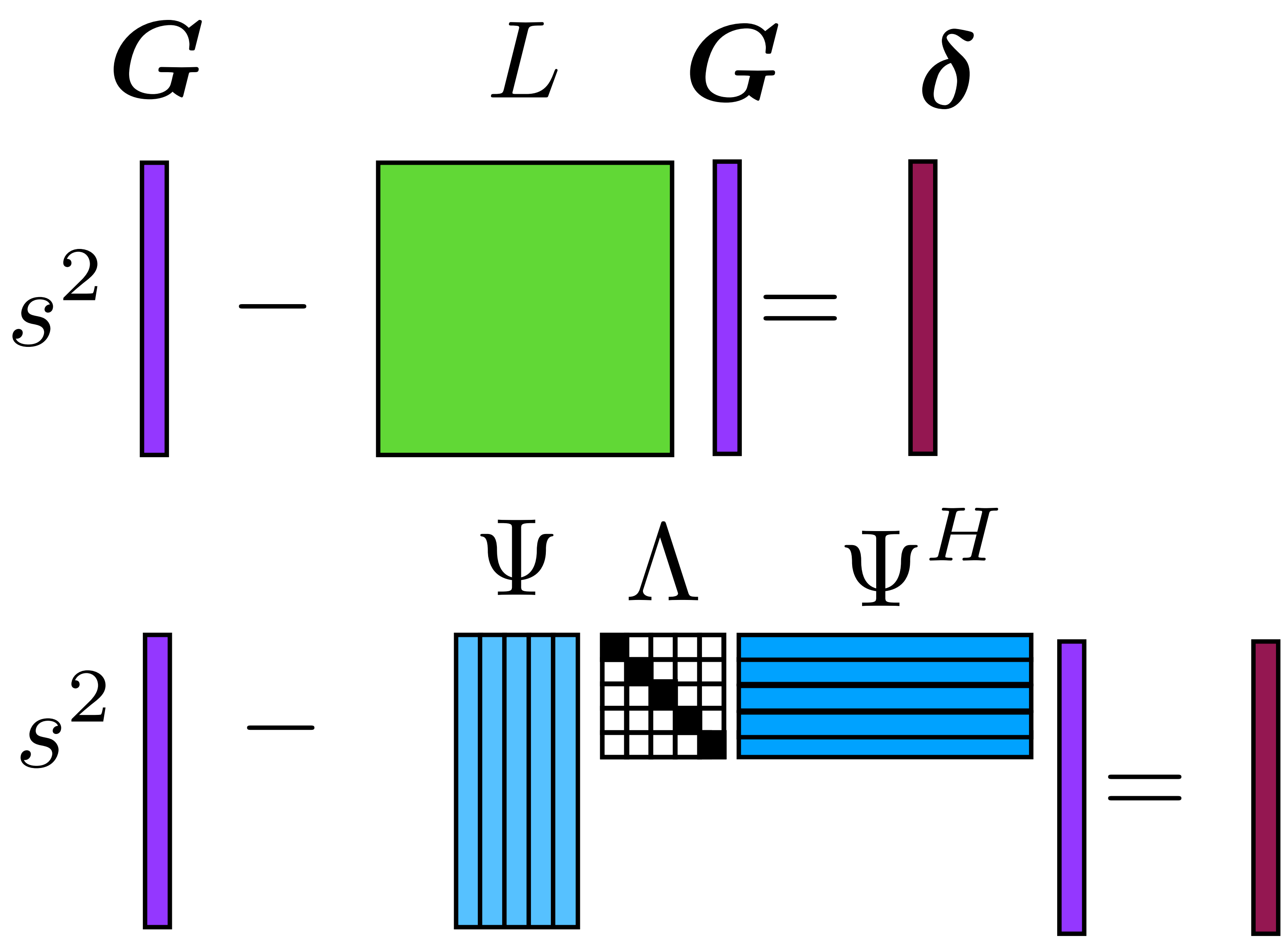
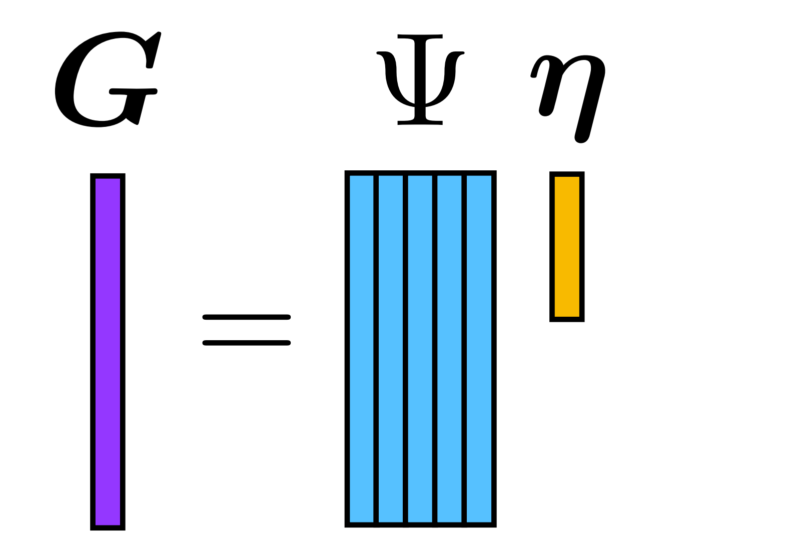


$$\boldsymbol{G} = \sum_{k=1}^{N} \eta_k \boldsymbol{\psi}_k = \Psi\boldsymbol{\eta}$$



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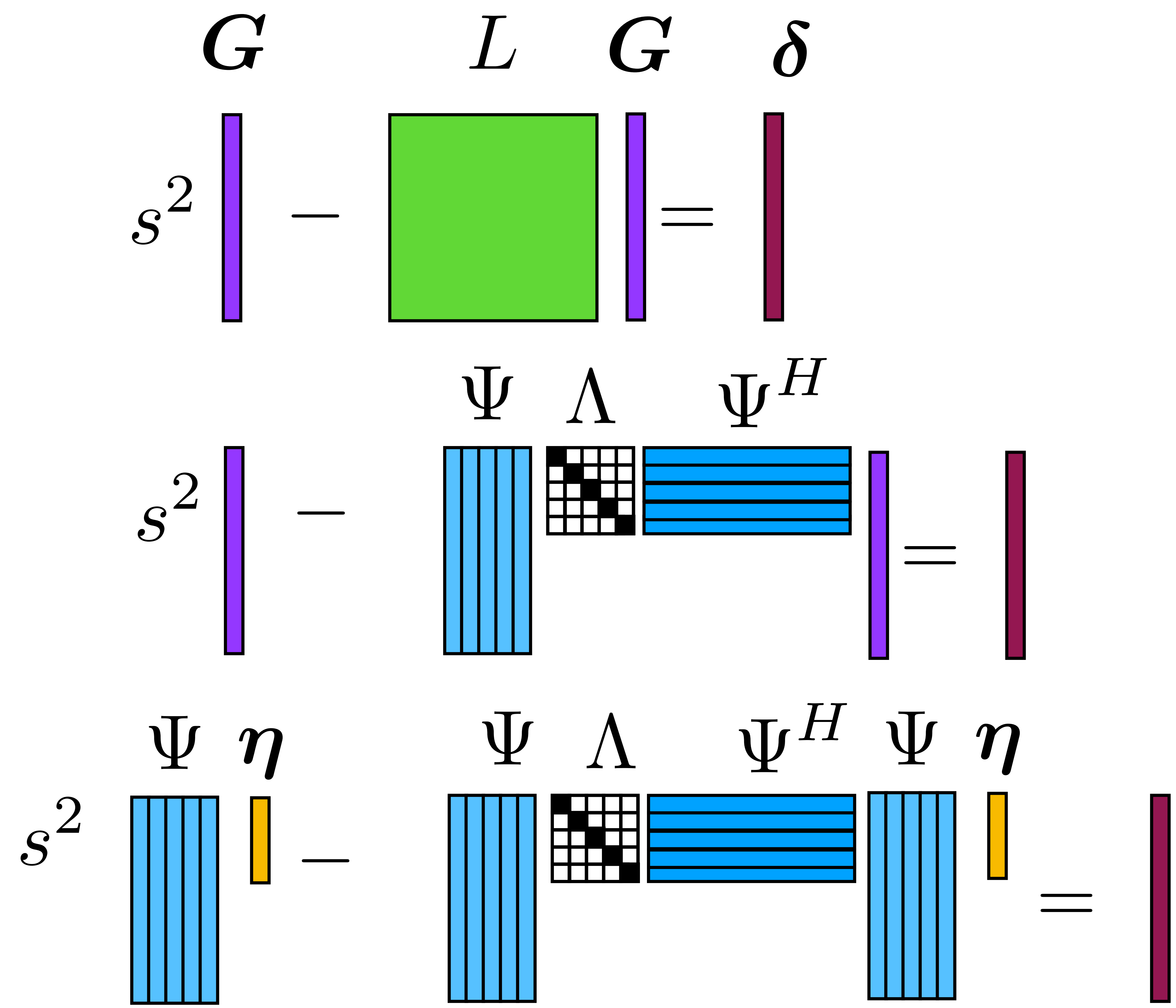

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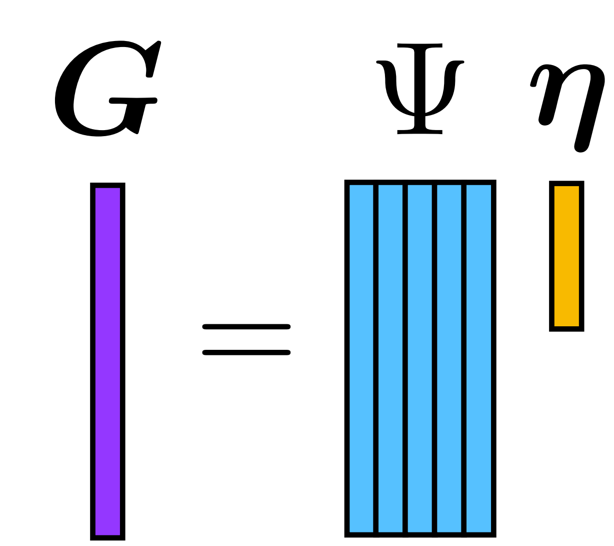

We exploit now the orthogonality (orthonormality) of the eigenmodes

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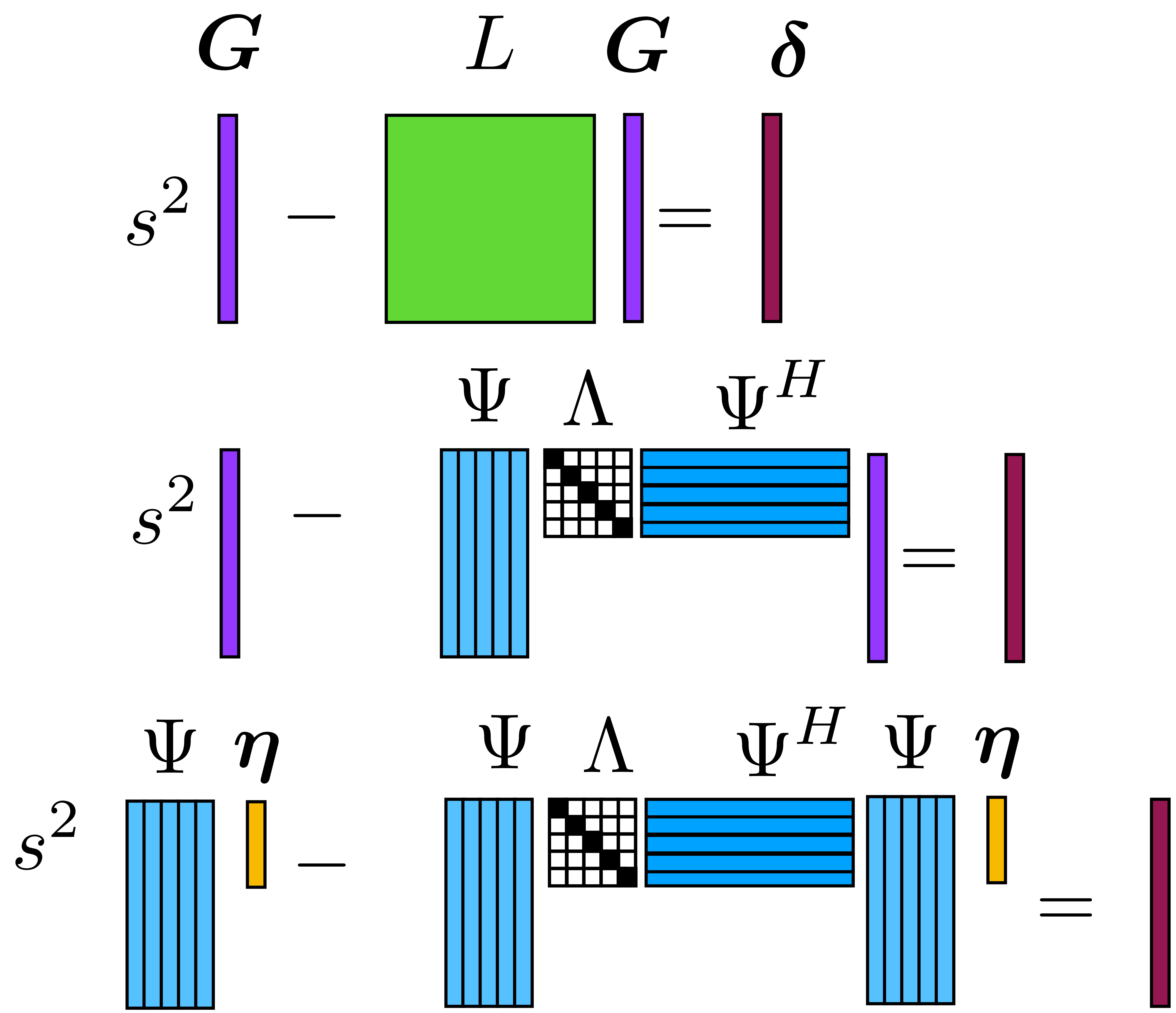

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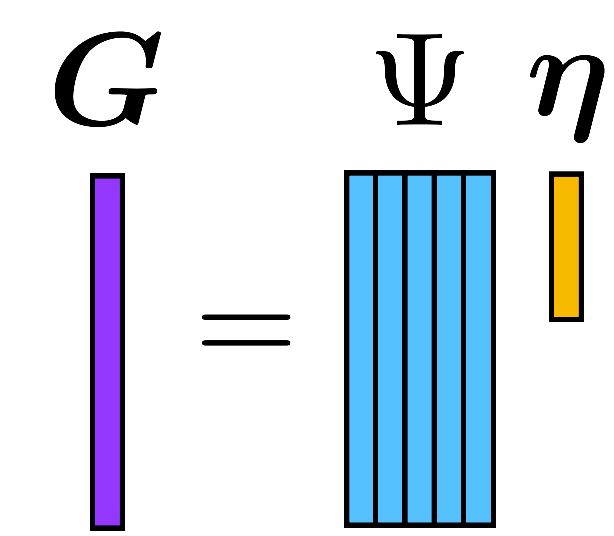

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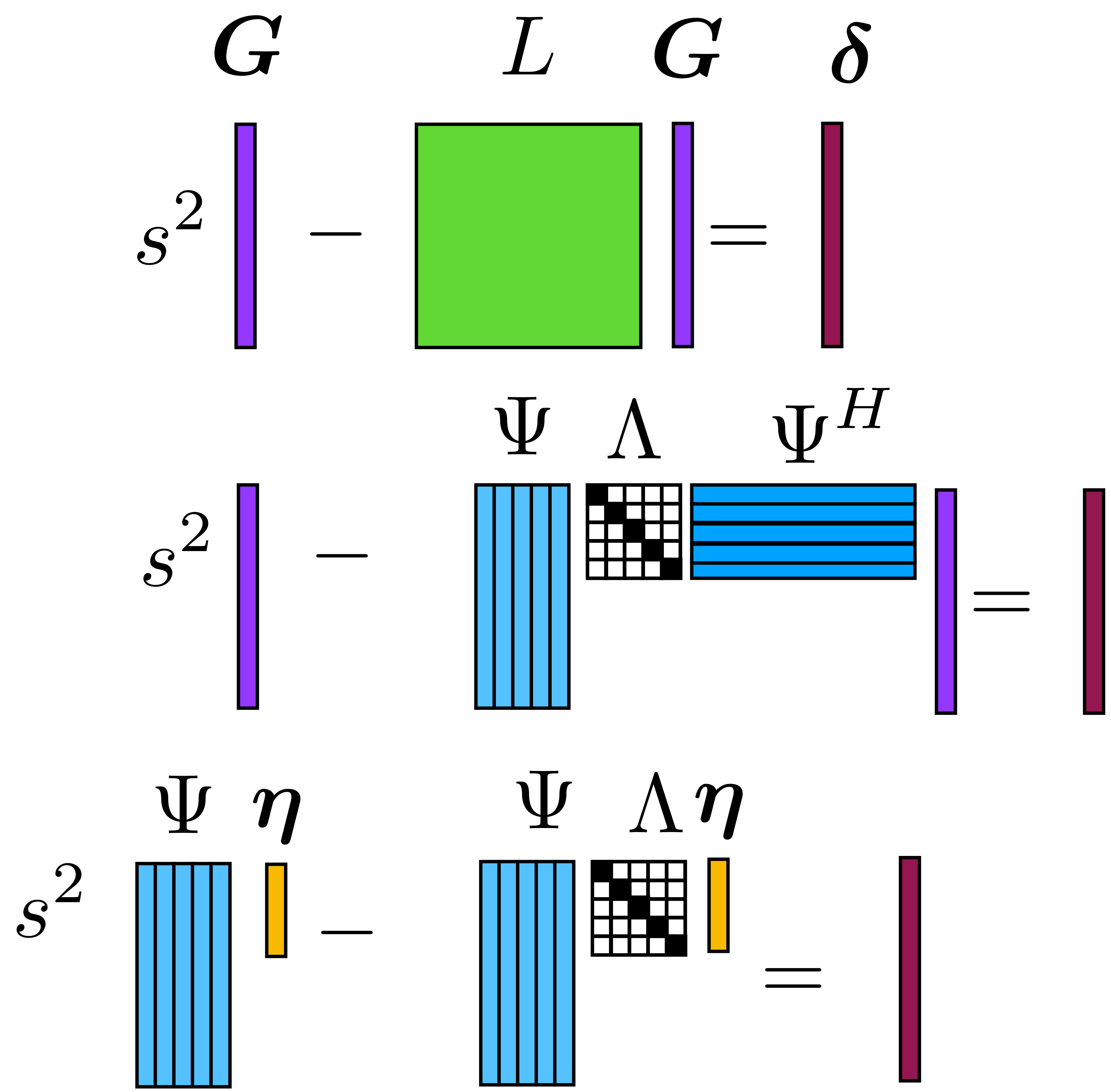

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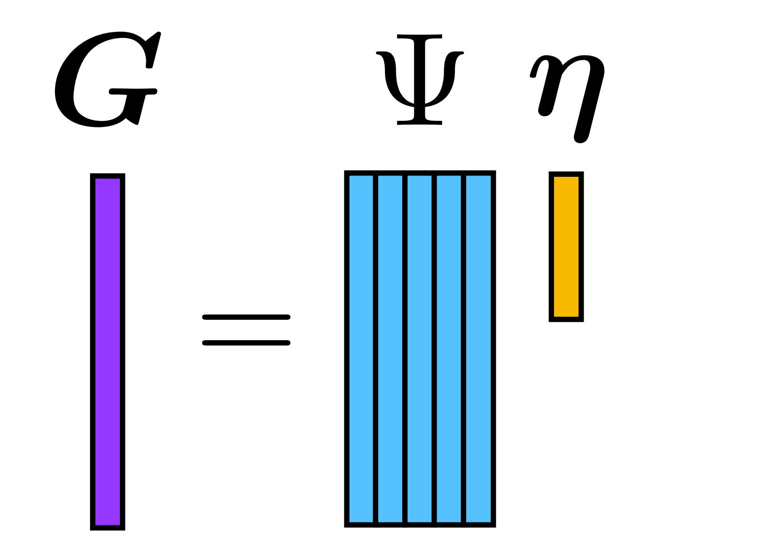

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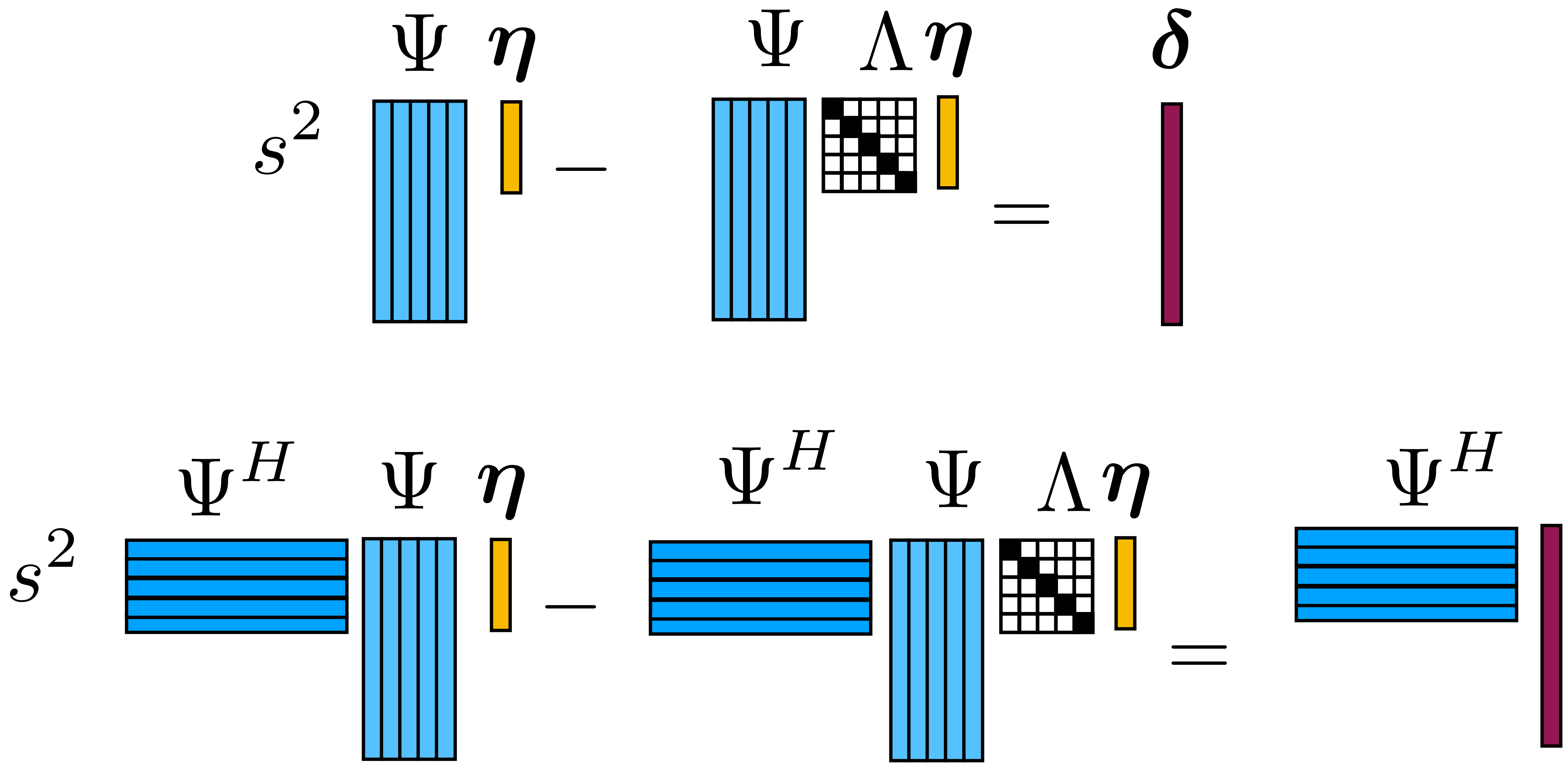

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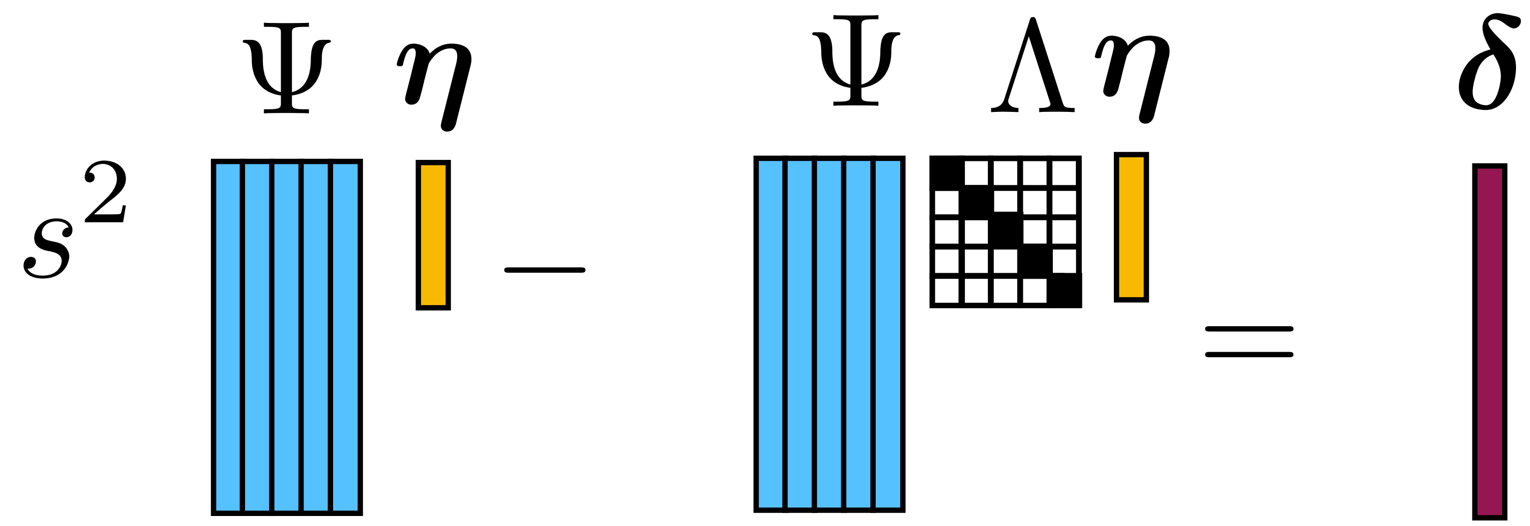

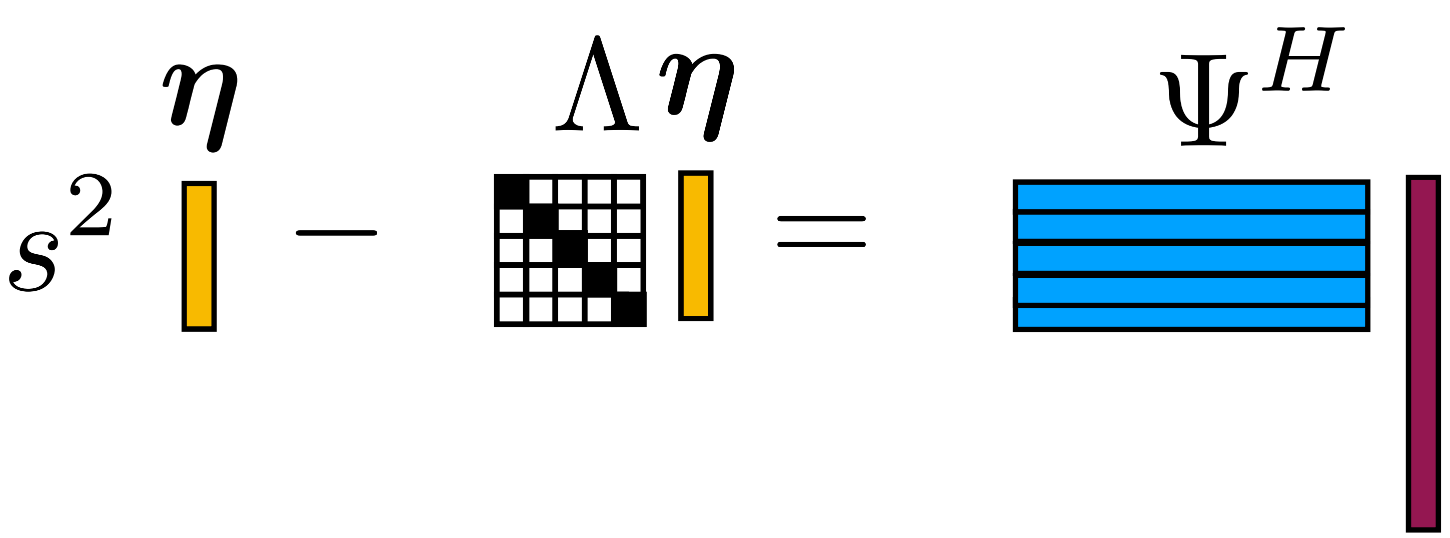

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$$s^2 \Psi \boldsymbol{\eta} - \Psi \Lambda \boldsymbol{\eta} = \boldsymbol{\delta}$$

$$s^2 \boldsymbol{\eta} - \Lambda \boldsymbol{\eta} = \Psi^H \boldsymbol{\delta} \quad \Rightarrow \quad \boldsymbol{\eta} = \left(I s^2 - \Lambda \right)^{-1} \Psi^H \boldsymbol{\delta}$$

By doing some linear algebra we have solved the Equation!!

$$s^2 \Psi \eta - \Psi \Lambda \eta = \delta$$

$$s^2 \eta - \Lambda \eta = \Psi^H \delta \quad \Rightarrow \quad \eta = \left(I s^2 - \Lambda \right)^{-1} \Psi^H \delta$$

recall that

$$G = \Psi \eta \quad \Rightarrow \quad G = \left(I s^2 - \Lambda \right)^{-1} \Psi \Psi^H \delta$$

The solution of the Helmholtz equation reads

Helmholtz Equation

$$s^2 \hat{p} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right)}_{\mathcal{L}\hat{p}} = \hat{h}$$

Boundary Conditions

$$\mathbf{n} \cdot \frac{\partial \hat{p}}{\partial x_i} = -\hat{f}$$

Solution

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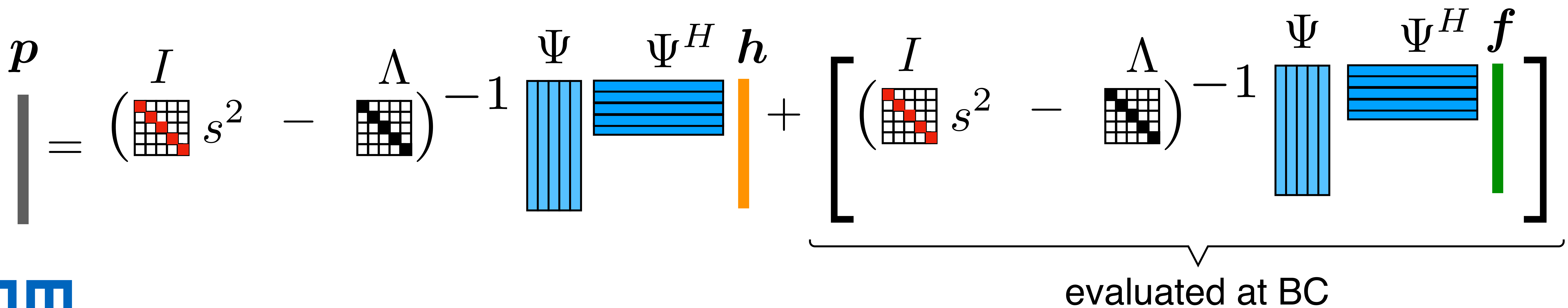


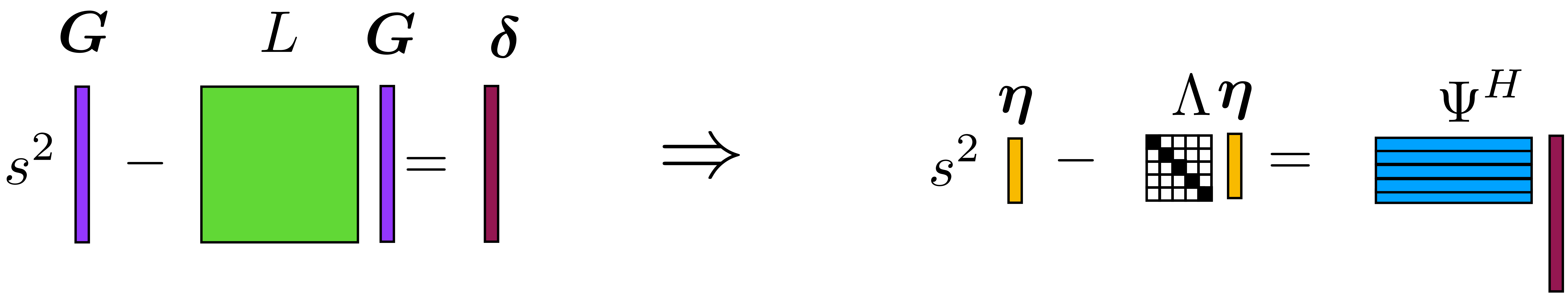
what about the state space approach?

Outline

- Solving the Helmholtz Equation by modal expansion
- And the state space?
- About a one mode expansion

We can recover a differential equation for η

After discretization of $$s^2\hat{G} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{G}}{\partial x_i}\right)}_{\mathcal{L}\hat{G}} = \delta(x - x_0)$$



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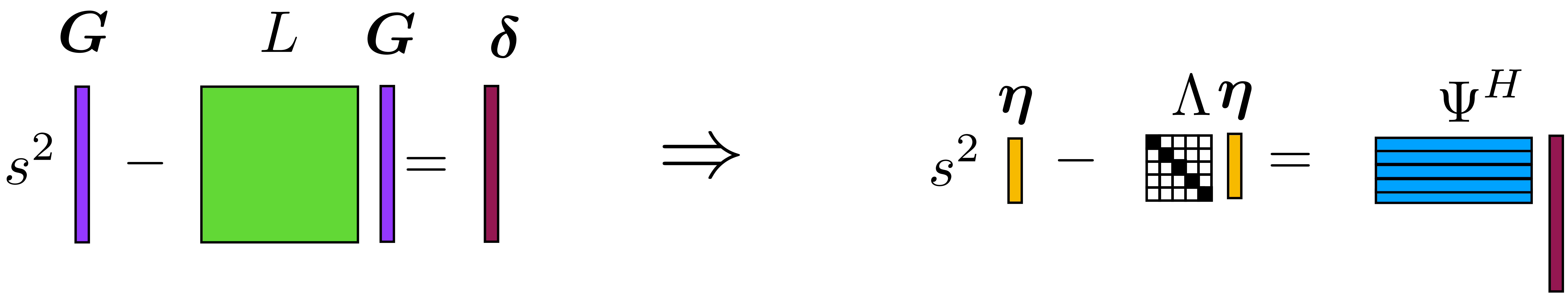

The dynamics of η is governed by

$$\ddot{\boldsymbol{\eta}} - \Lambda\boldsymbol{\eta} = \Psi^H\boldsymbol{\delta}$$

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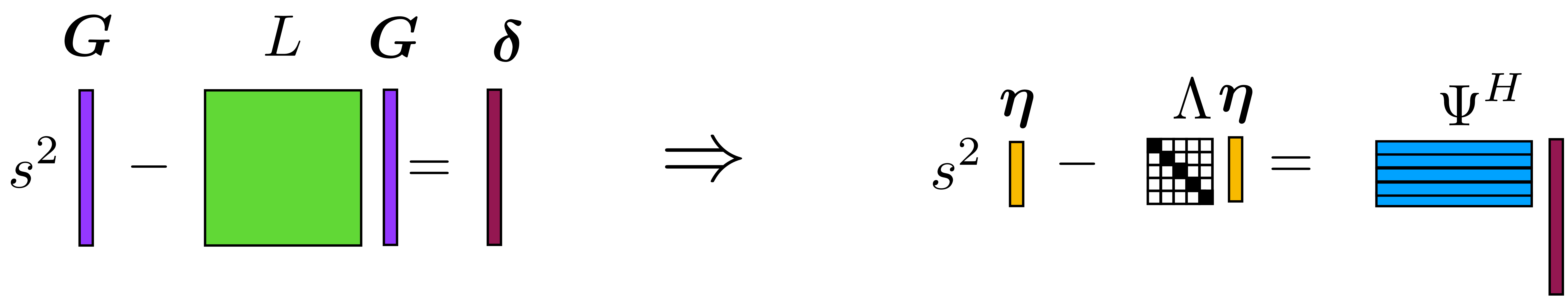

The dynamics of η is governed by

$$\ddot{\boldsymbol{\eta}} - \Lambda\boldsymbol{\eta} = \Psi^H\boldsymbol{\delta} \quad \Rightarrow \quad \begin{aligned} \dot{\boldsymbol{\eta}} &= \boldsymbol{\upsilon} \\ \dot{\boldsymbol{\upsilon}} &= \Lambda\boldsymbol{\eta} + \Psi^H\boldsymbol{\delta} \end{aligned}$$

↑ state vector

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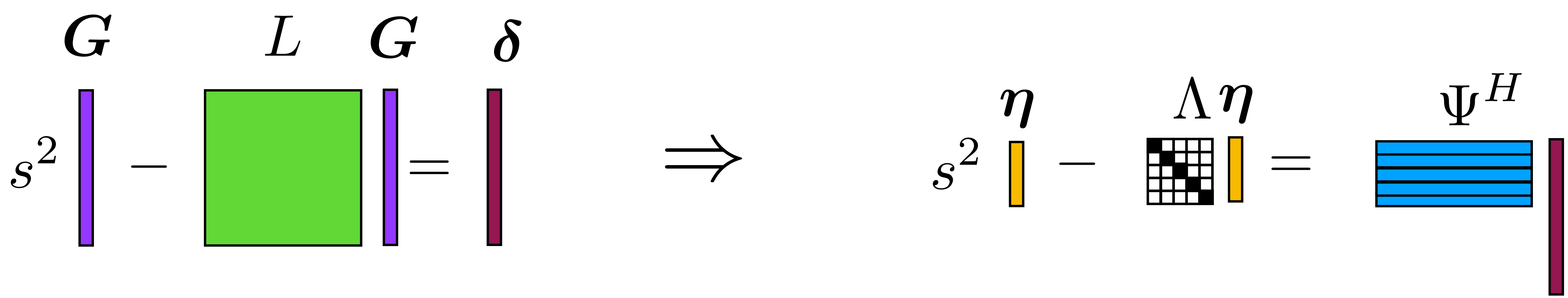

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These Equations are the first step towards the state space representation

Does makes to consider only one mode in the expansion?

Yes!

Outline

- Solving the Helmholtz Equation by modal expansion

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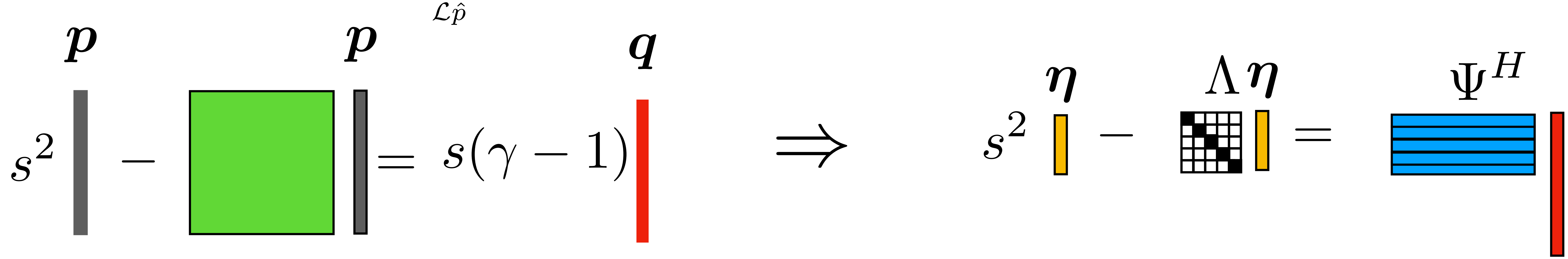
- About a one mode expansion

Let us assume homogeneous Neumann BC (no flux)

Solution of $$s^2\hat{p} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2\frac{\partial \hat{p}}{\partial x_i}\right)}_{\mathcal{L}\hat{p}} = s(\gamma-1)\hat{\dot{q}}$$



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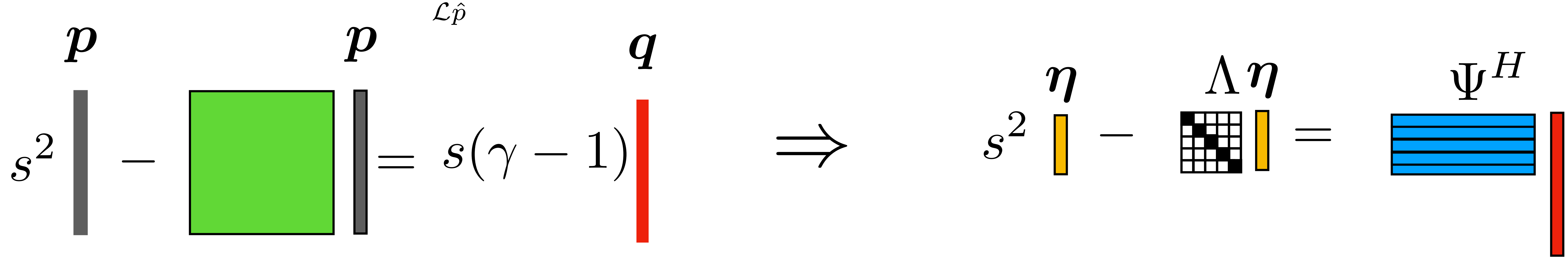

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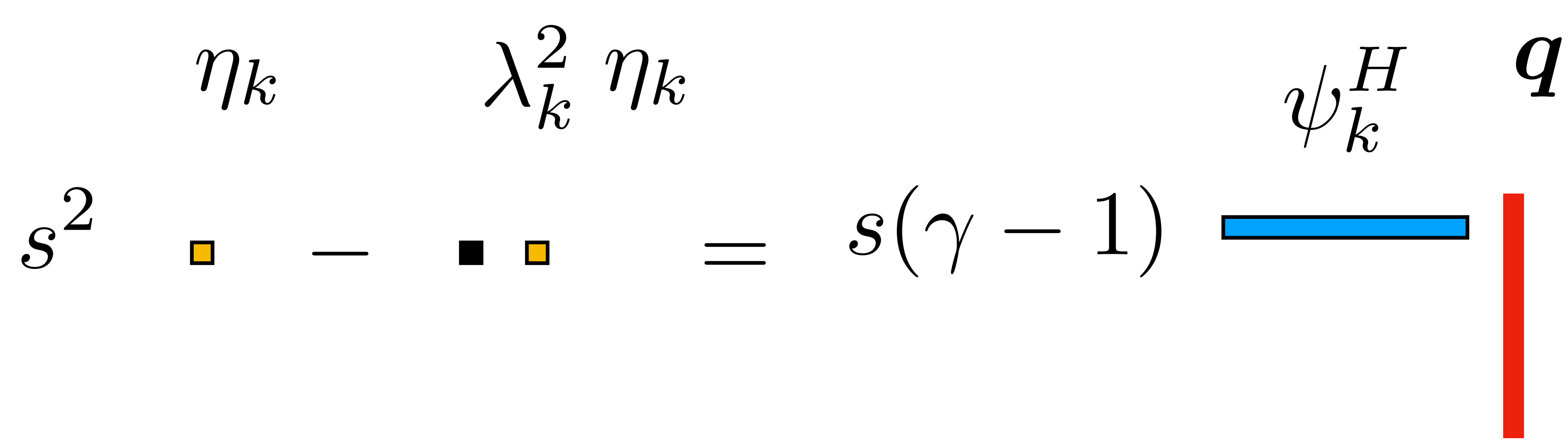

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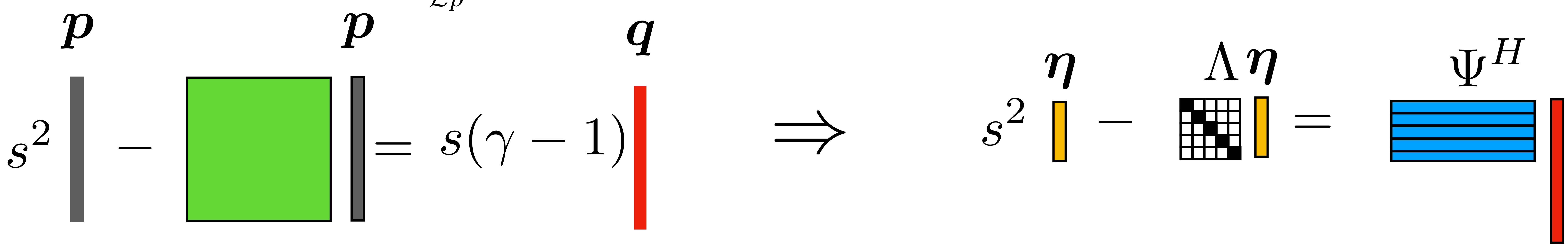

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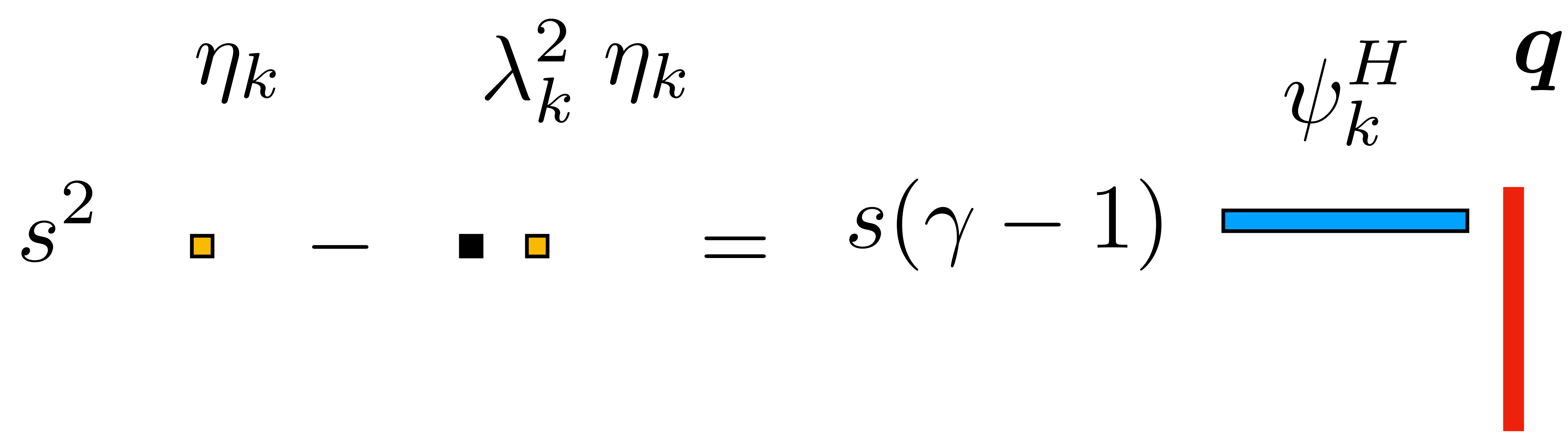

$$s^2\eta_k - \lambda_k^2\eta_k = s(\gamma - 1)\boldsymbol{\psi}_k^H \boldsymbol{q}$$

Let us assume homogeneous Neumann BC (no flux)

We have gone from here

$$s^2\hat{p} - \underbrace{\frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right)}_{\mathcal{L}\hat{p}} = s(\gamma - 1)\hat{\dot{q}}$$

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Consider now two models for the flame response

model 1

$$\boldsymbol{q} = I_{\mathrm{f}} I_{\mathrm{ref}}^{\mathsf{T}} \boldsymbol{p}\; n_{\mathrm{p}} e^{-s\tau_{\mathrm{p}}} \approx \eta_k I_{\mathrm{f}} I_{\mathrm{ref}}^{\mathsf{T}} \boldsymbol{\psi}_k n_{\mathrm{p}} e^{-s\tau_{\mathrm{p}}}$$

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Consider now two models for the flame response

model 1

$$\boldsymbol{q} = I_\mathrm{f}I_\mathrm{ref}^\intercal\boldsymbol{p}\; n_\mathrm{p}e^{-s\tau_\mathrm{p}} \approx \eta_k I_\mathrm{f}I_\mathrm{ref}^\intercal\boldsymbol{\psi}_k n_\mathrm{p}e^{-s\tau_\mathrm{p}} \quad \Longrightarrow \quad s^2 - \lambda_k^2 = s(\gamma-1)\kappa n_\mathrm{p}e^{-s\tau_\mathrm{p}}$$

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Let us assume homogeneous Neumann BC (no flux)

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The entire PDE has been reduced to a simple algebraic equation !

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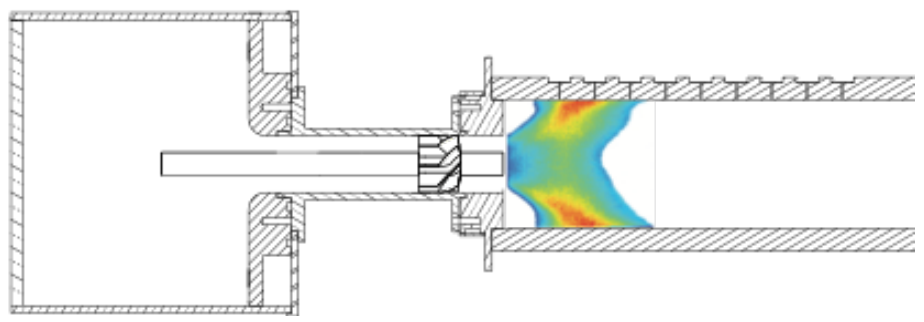
For the case investigated, the agreement of the two approaches is remarkable

system under study



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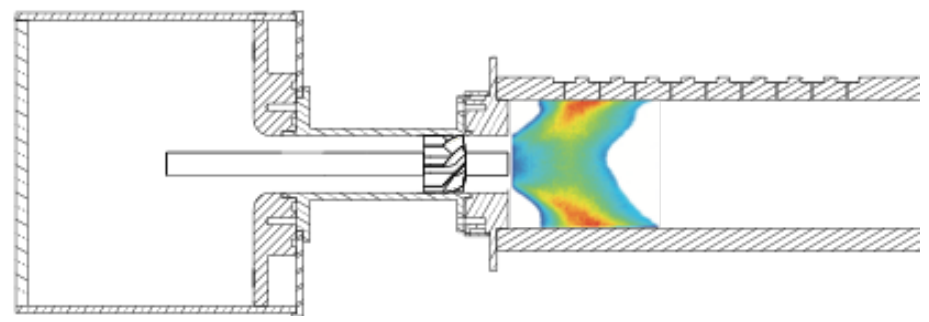

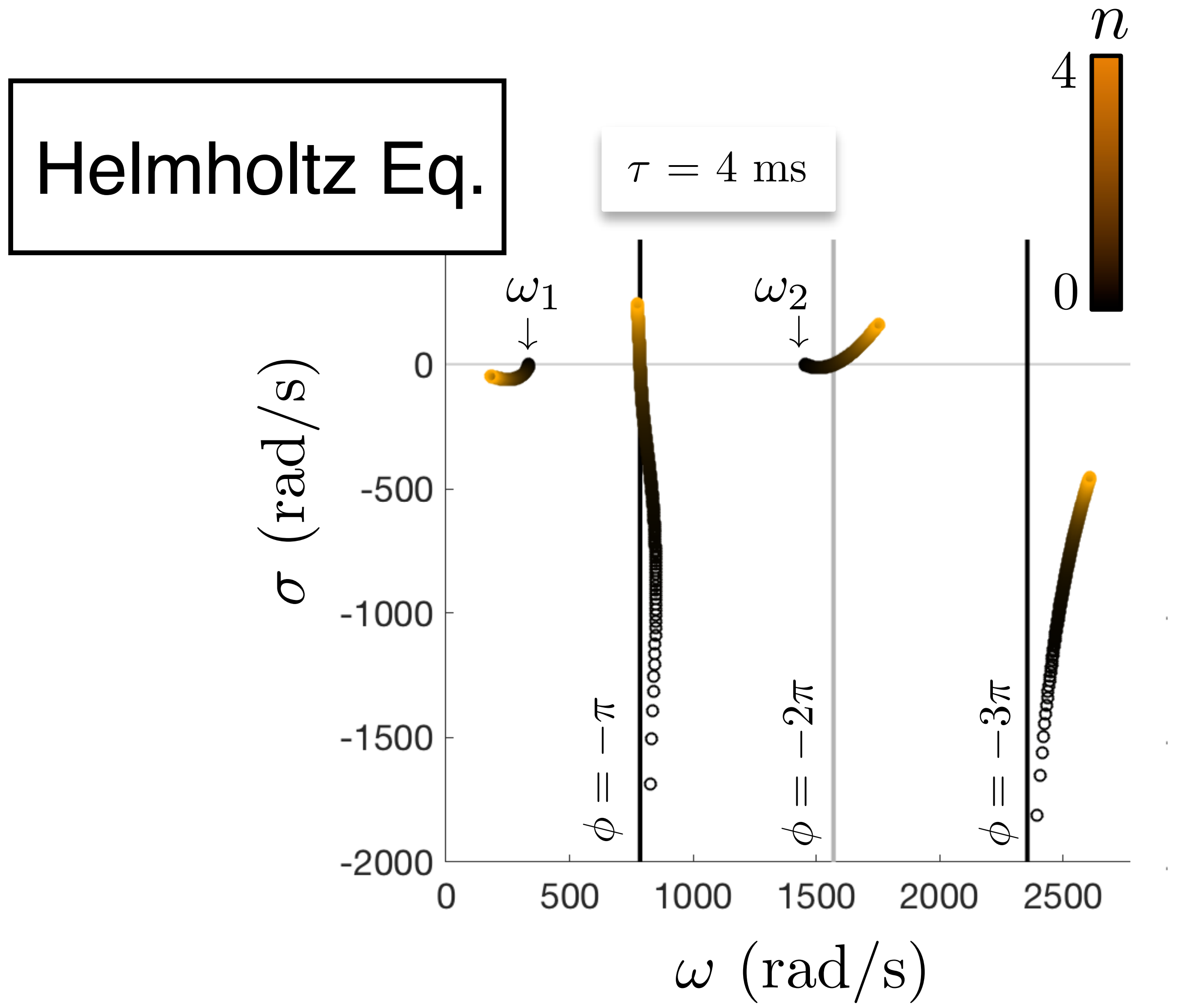

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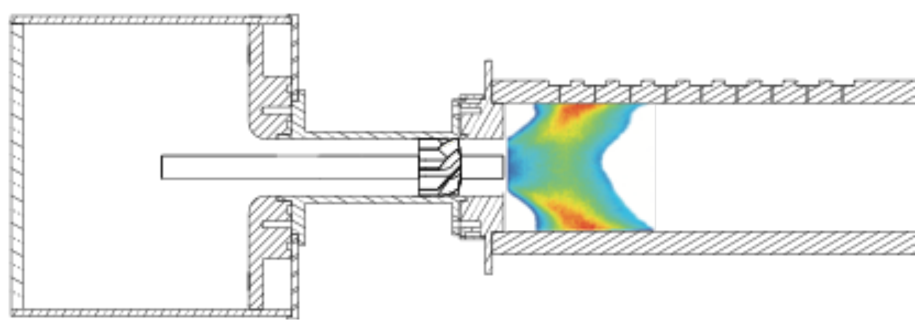

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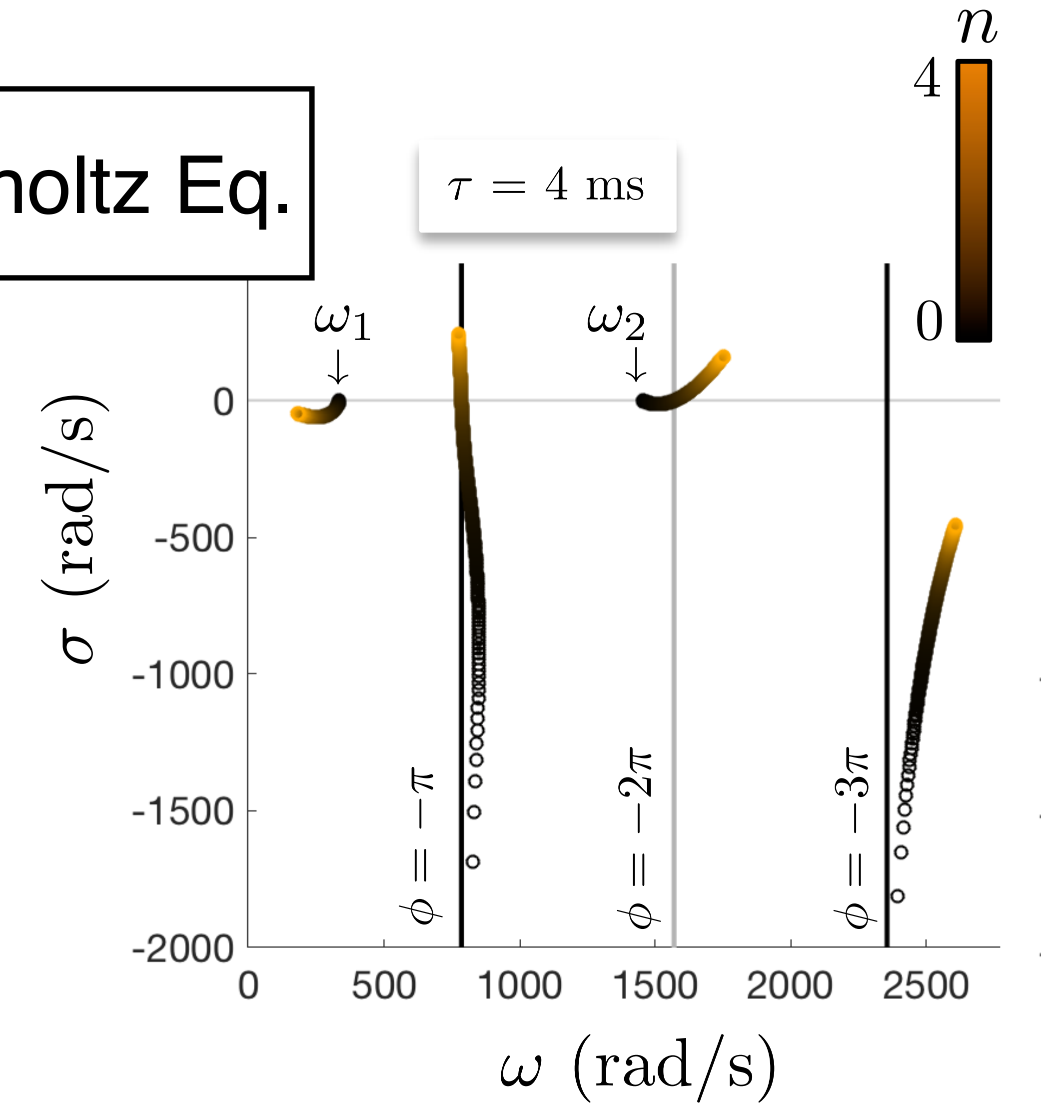

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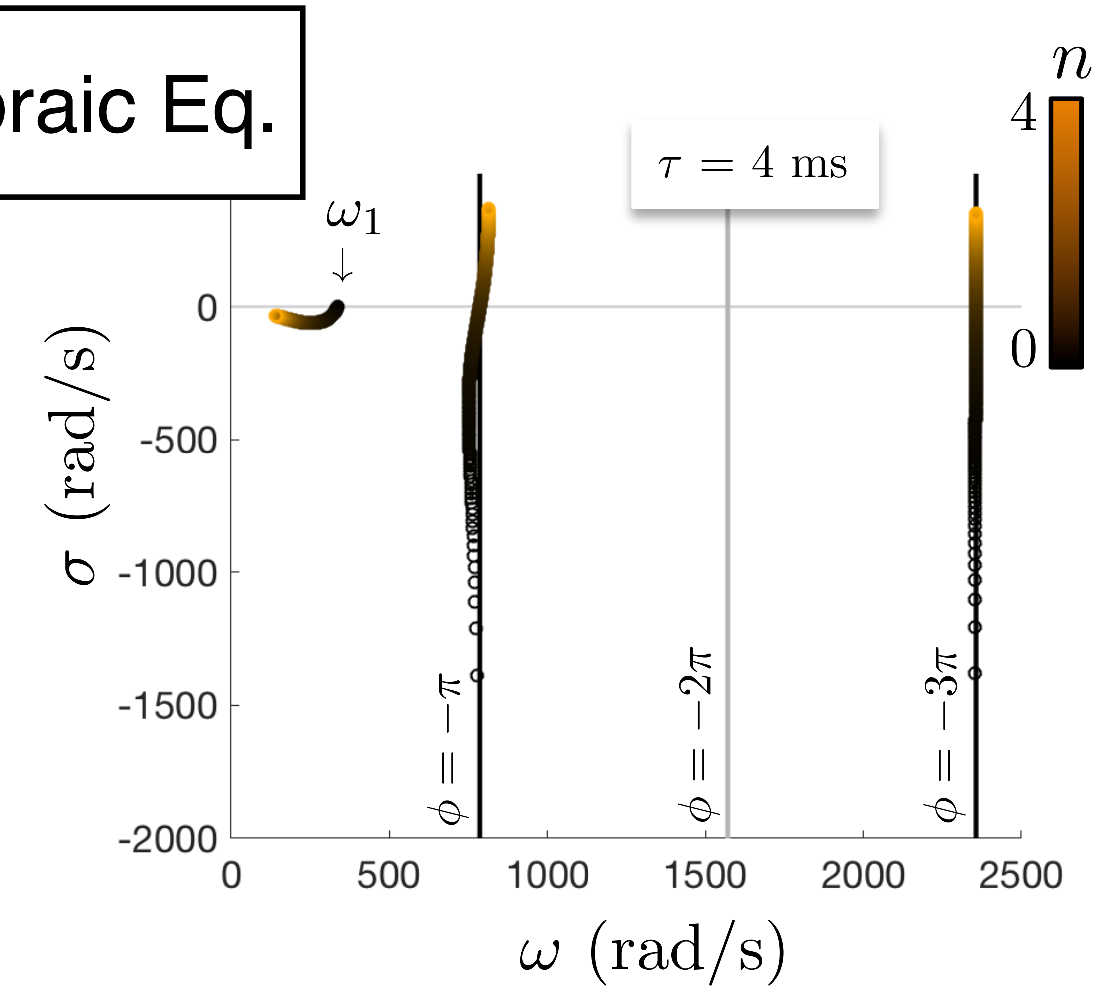

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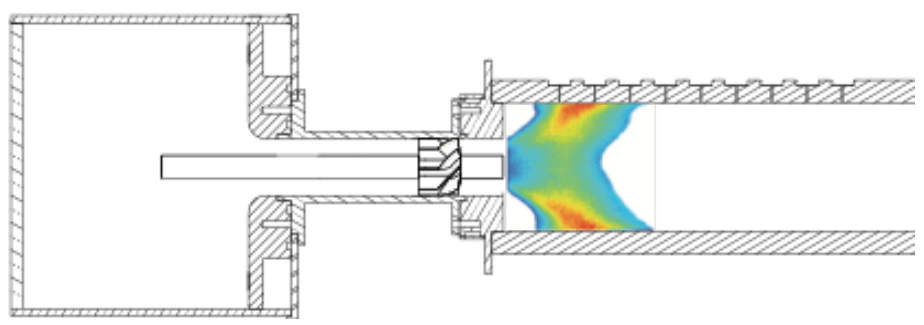

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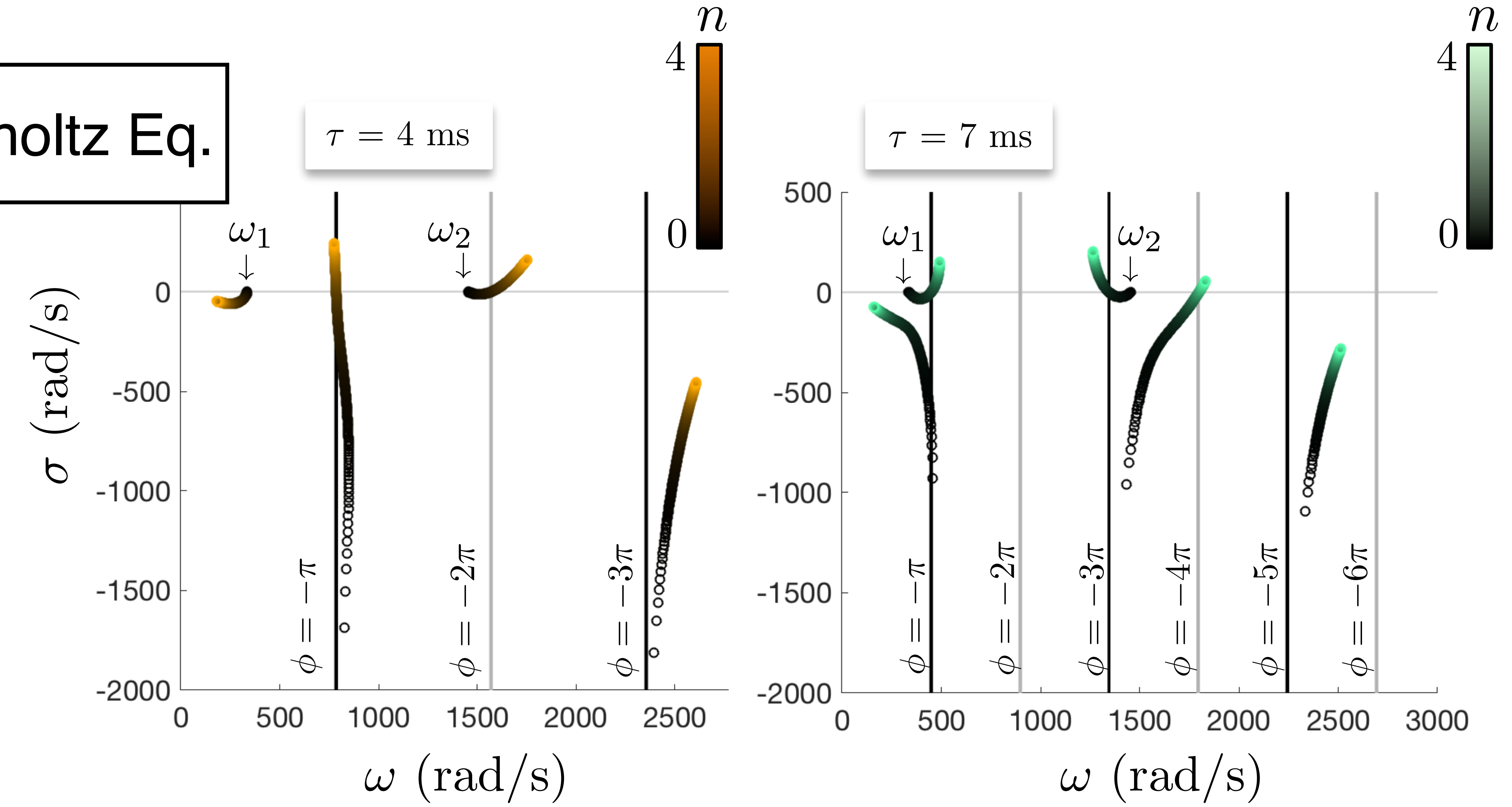

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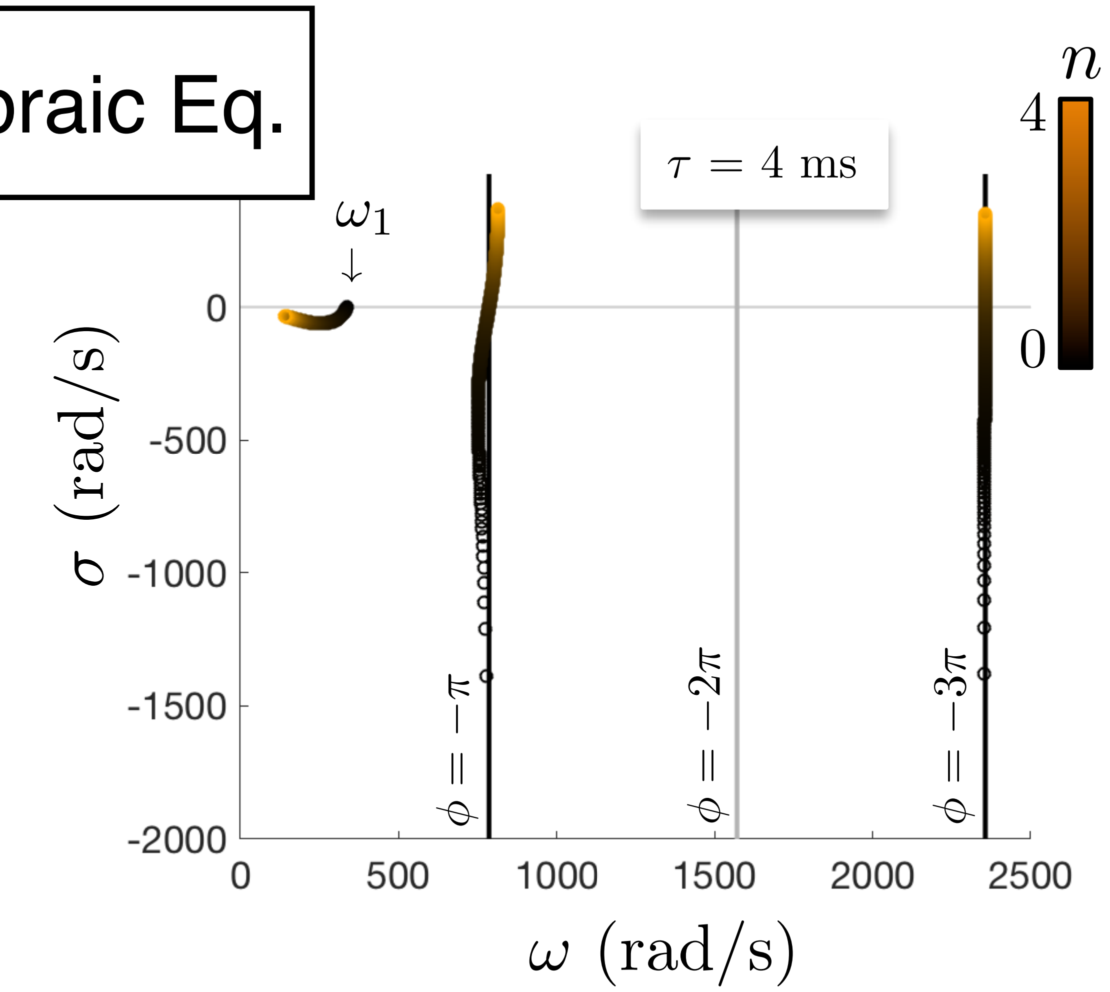

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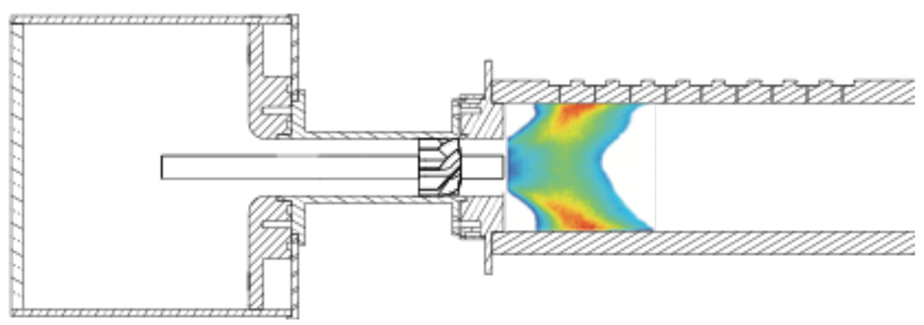

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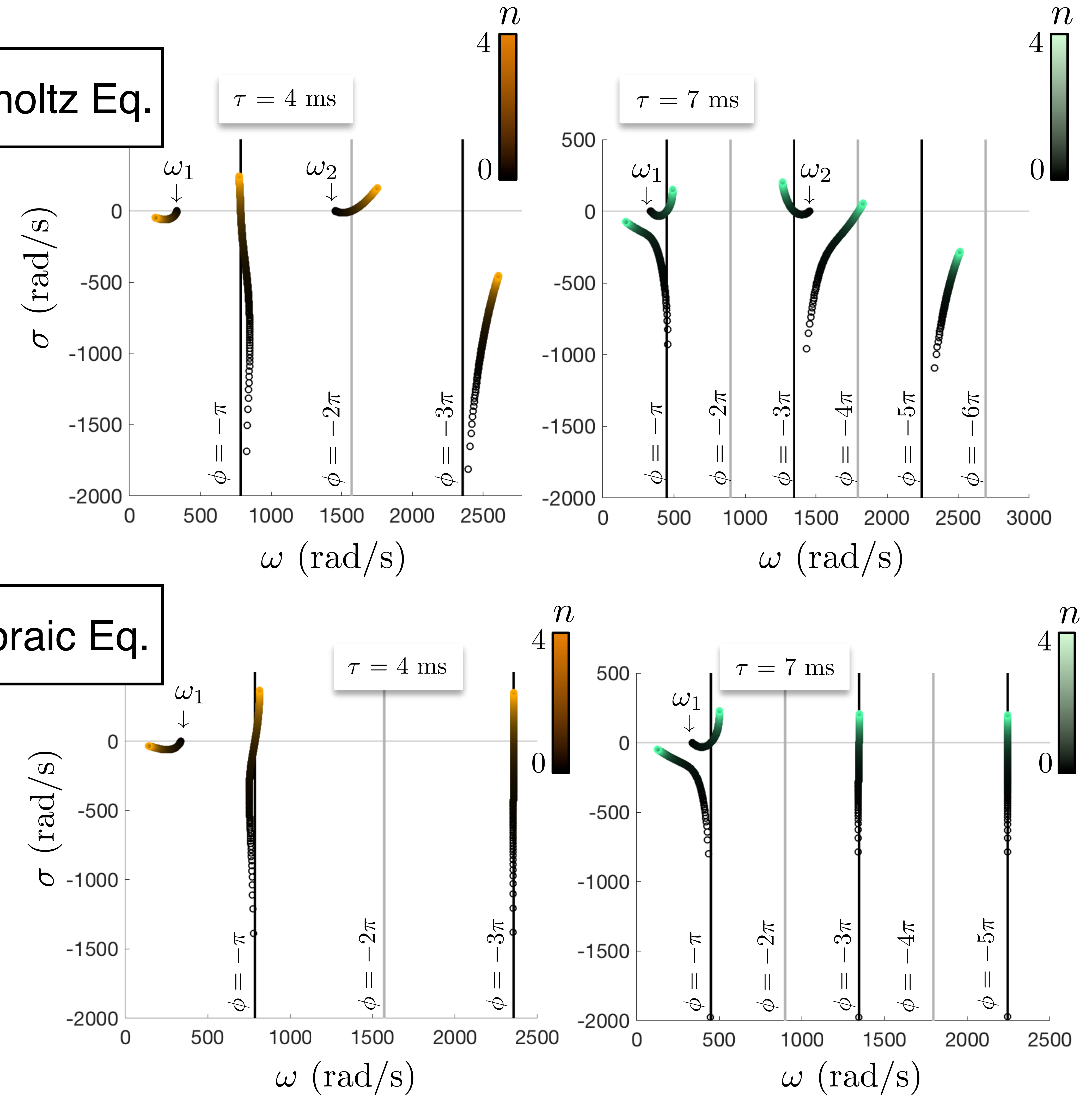

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