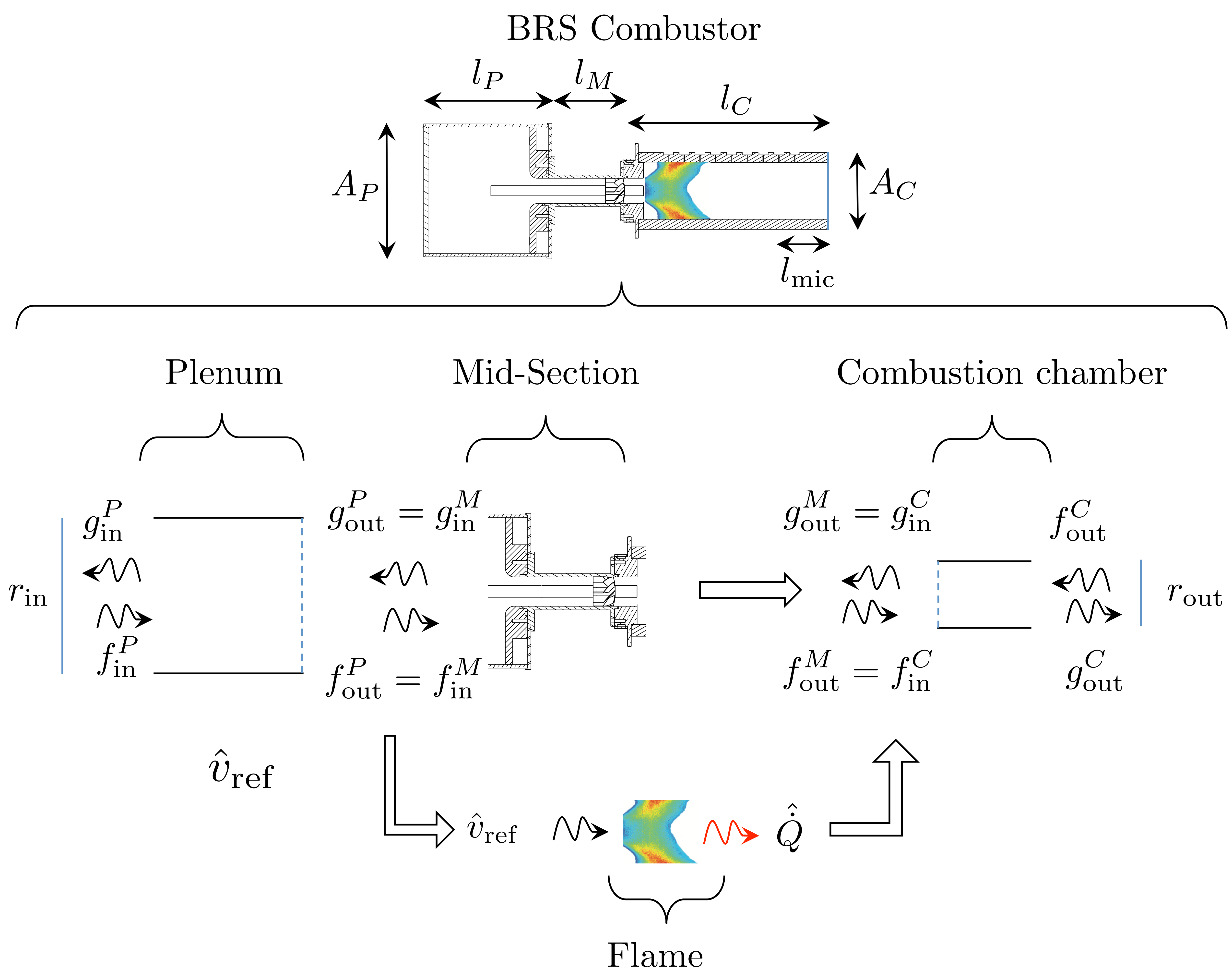
Generalities of thermoacoustic network models

Camilo F. Silva

April 27, 2022

And what if we want to apply different models for different parts?

A proper network model should be able to integrate information coming from different kind of models and external data



Outline

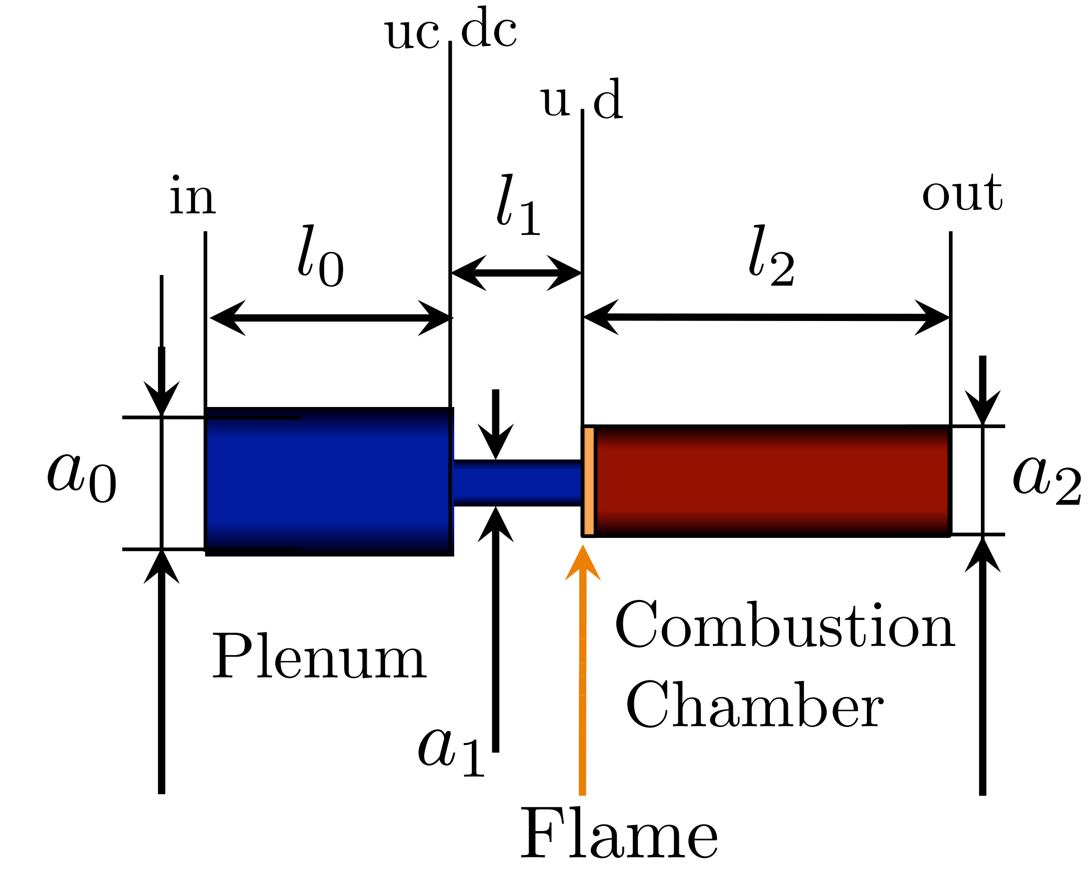
- From the Navier-Stokes equations to the acoustic jump conditions
- From primitive variables to acoustic invariants (waves)
- The state space approach

Jump conditions in a duct with changes in area, temperature and including the flame response can be done via conservation equations

From the previous lecture recall that

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{1}{\gamma p}\frac{Dp}{Dt} + \frac{\partial u_i}{\partial x_i} = \frac{(\gamma - 1)}{\gamma p}\dot{q}$$



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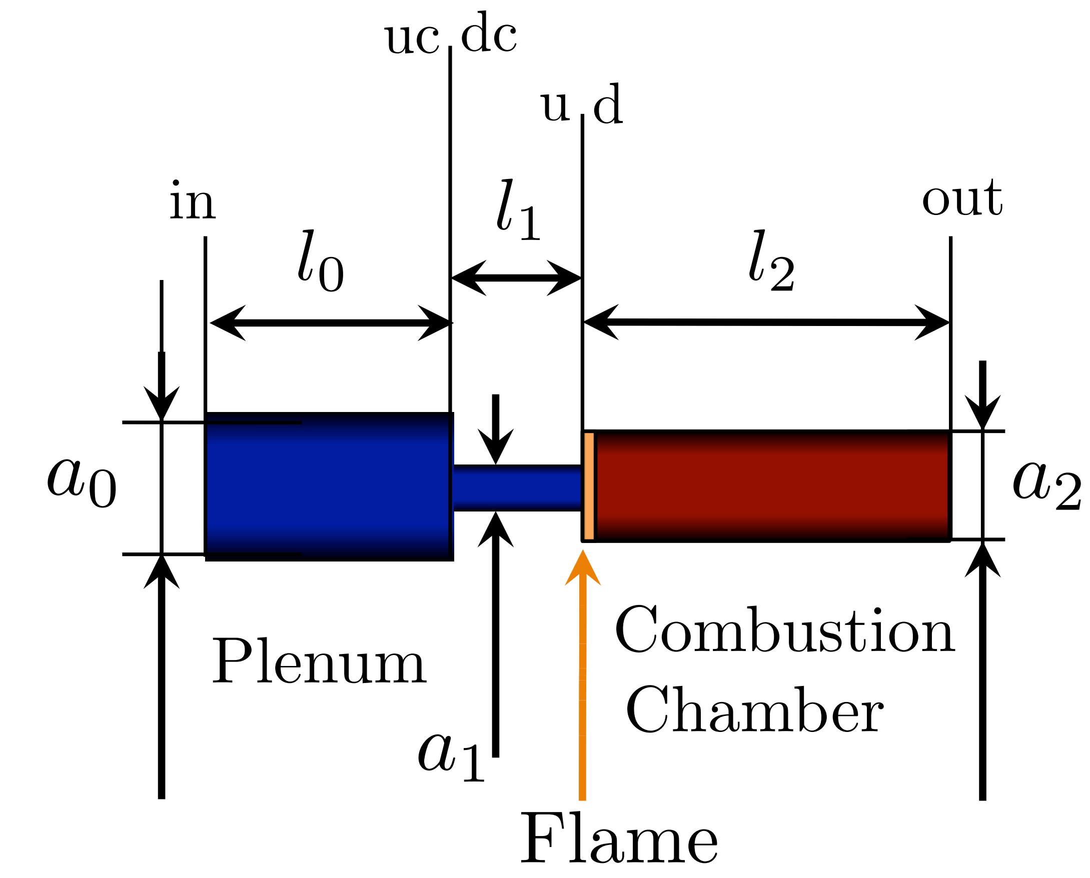

By assuming a low-Mach number flow, neglecting viscous terms and linearizing, we obtain

$$\frac{\partial u_i'}{\partial t} = -\frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x_i}$$

$$\frac{1}{\gamma \bar{p}}\frac{\partial p'}{\partial t} + \frac{\partial u_i'}{\partial x_i} = \frac{(\gamma - 1)}{\gamma \bar{p}}\dot{q}'$$

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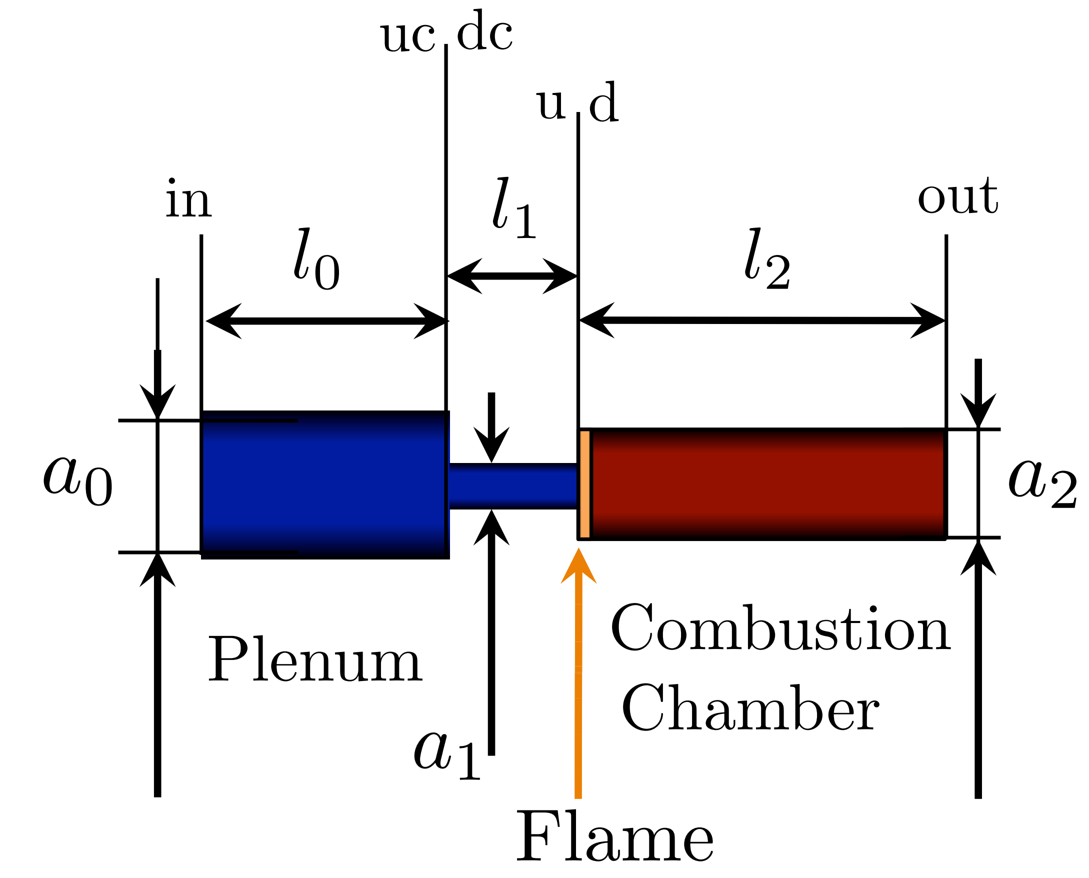

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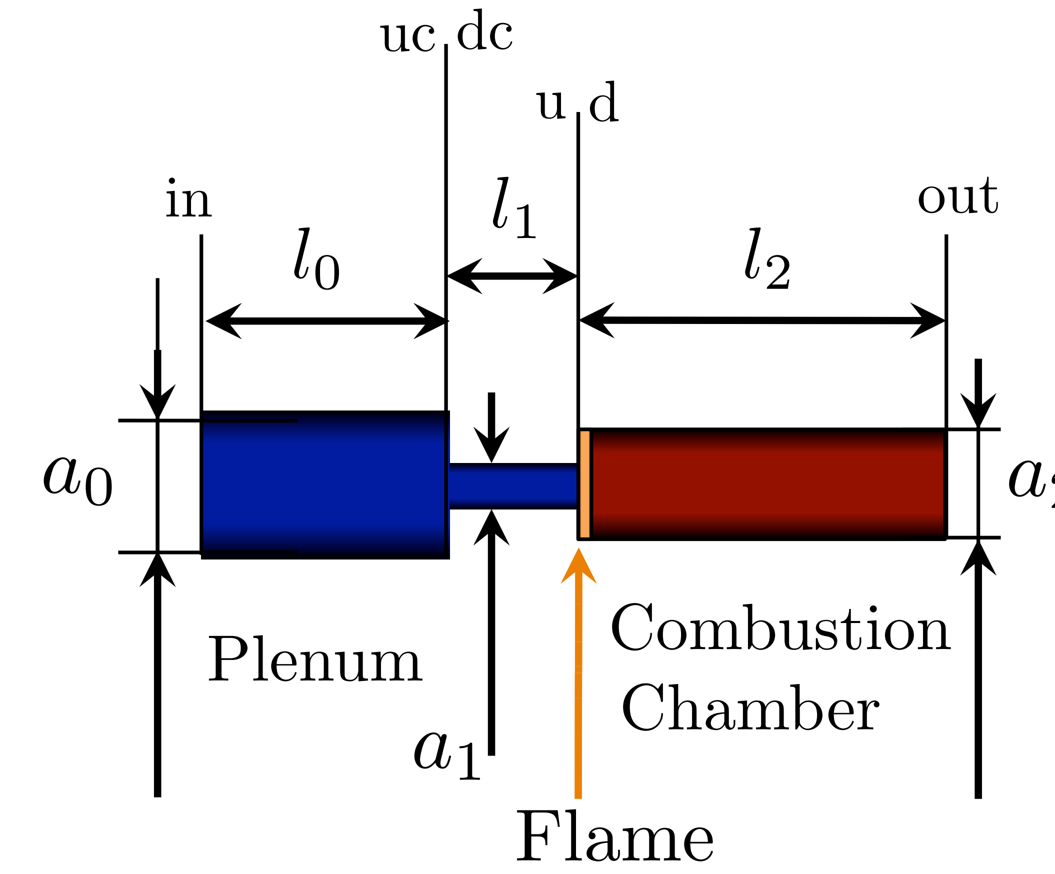

Consider now a quasi-1D flow. The above equations become

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Nicoud and Wieczorek 2009

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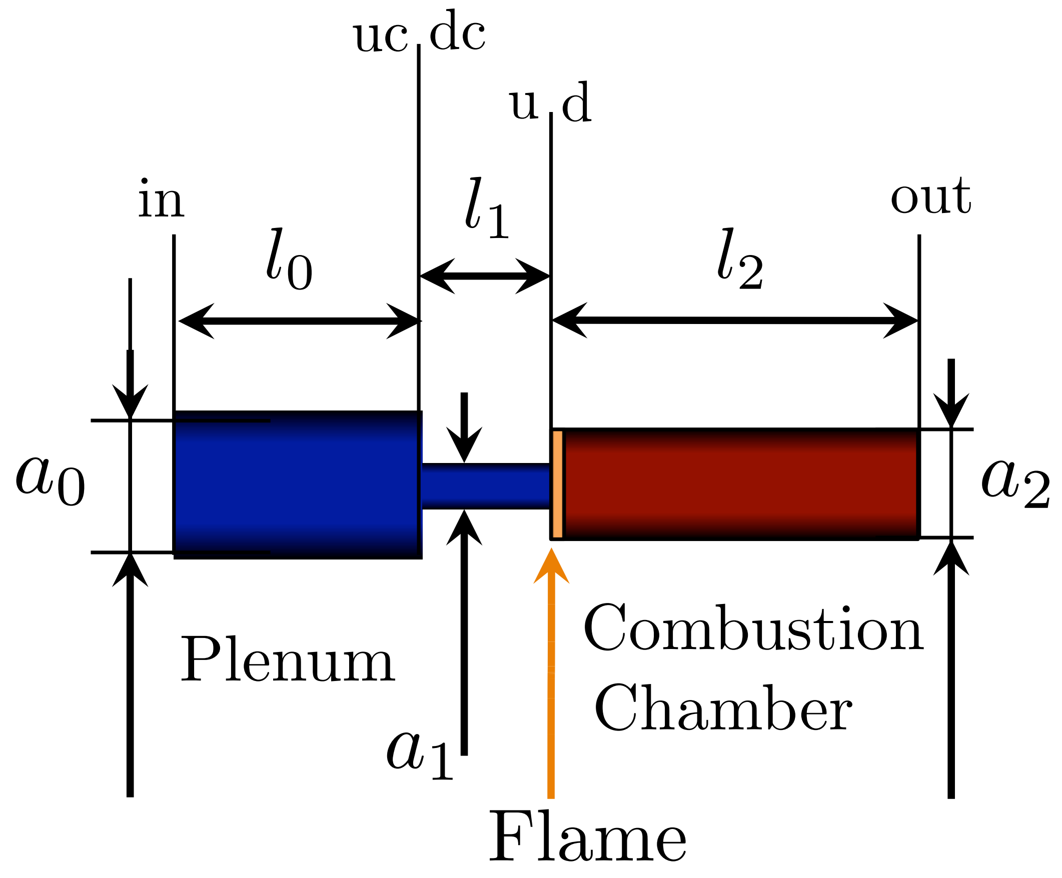

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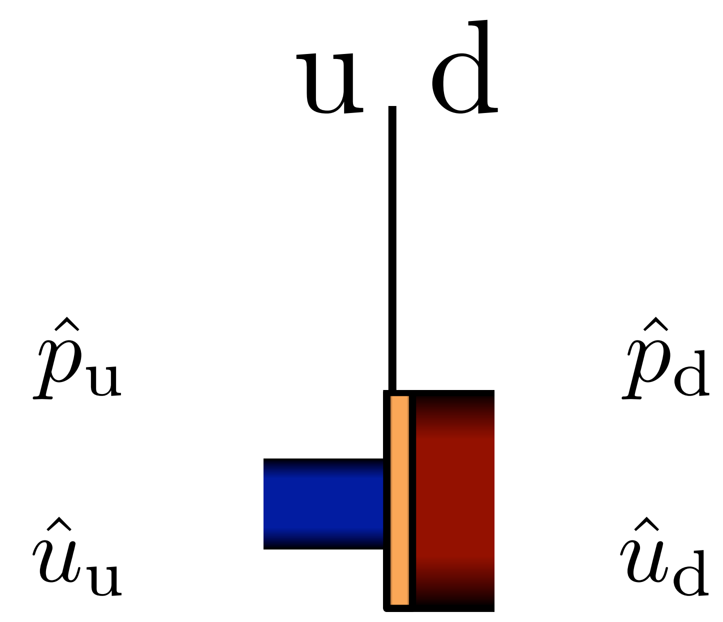
Finally we integrated in x the quasi-1D equations



$$\frac{\partial}{\partial t}\int_\mathrm{u}^\mathrm{d} \bar{\rho} u' \mathrm{d}x = -\left[p\right]_\mathrm{u}^\mathrm{d}$$

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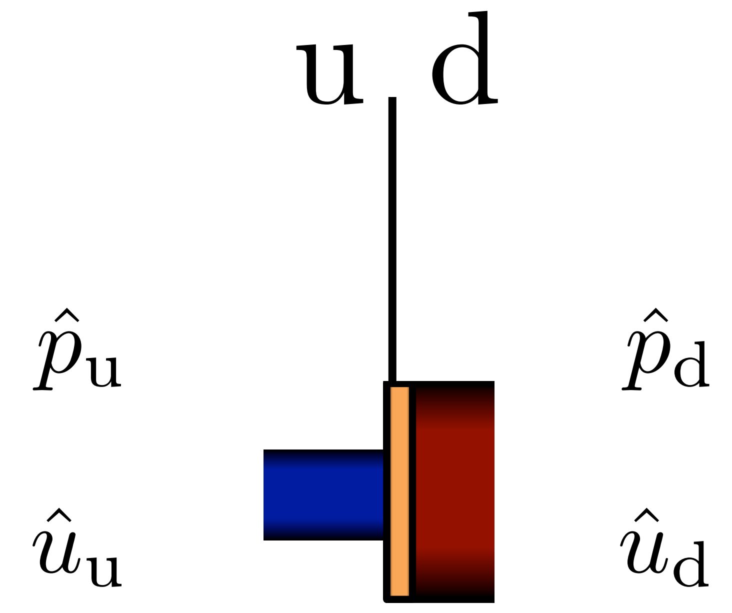

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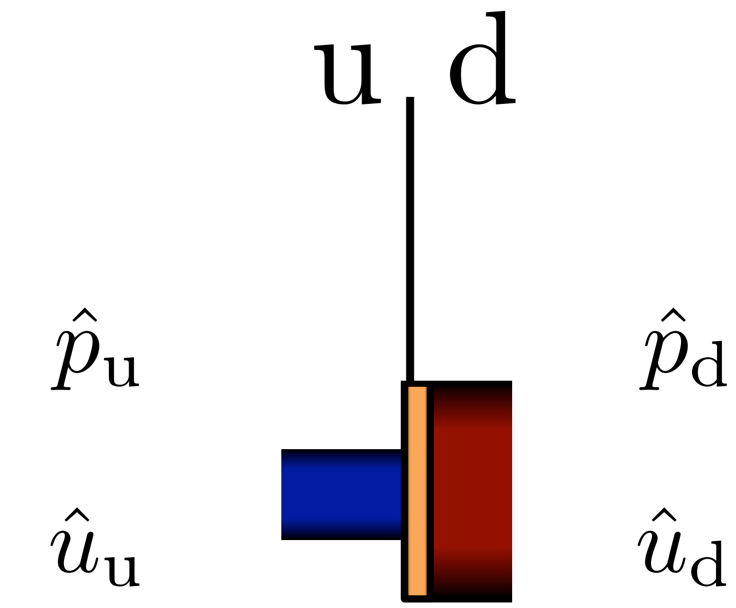
Applying the compact assumption and considering $[]' = \hat{[]} e^{st}$

$$\hat{p}_\mathrm{d} = \hat{p}_\mathrm{u}$$

$$a_\mathrm{d}\hat{u}_\mathrm{d} + a_\mathrm{u}\hat{u}_\mathrm{u} = \frac{(\gamma - 1)}{\gamma \bar{p}}\hat{\dot{Q}}$$

Note that we have neglected viscous terms. Their effect can be brought back by adding some terms in the derived relations

Finally we integrated in x the quasi-1D equations



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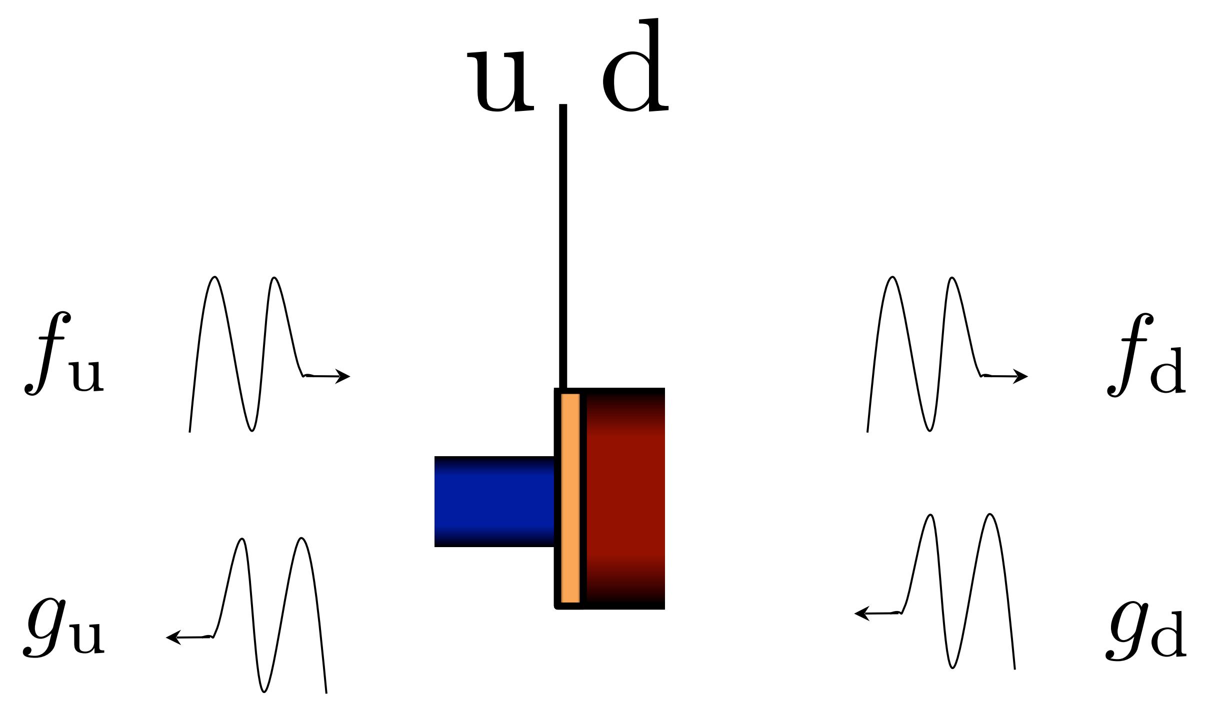
At this point it is of great interest to introduce the definition of acoustic waves

$$\hat{f} = \frac{1}{2}\left(\frac{\hat{p}}{\overline{\rho}\overline{c}} + \hat{u}\right) \quad \text{and} \quad \hat{g} = \frac{1}{2}\left(\frac{\hat{p}}{\overline{\rho}\overline{c}} - \hat{u}\right)$$

Downstream traveling wave — Upstream traveling wave



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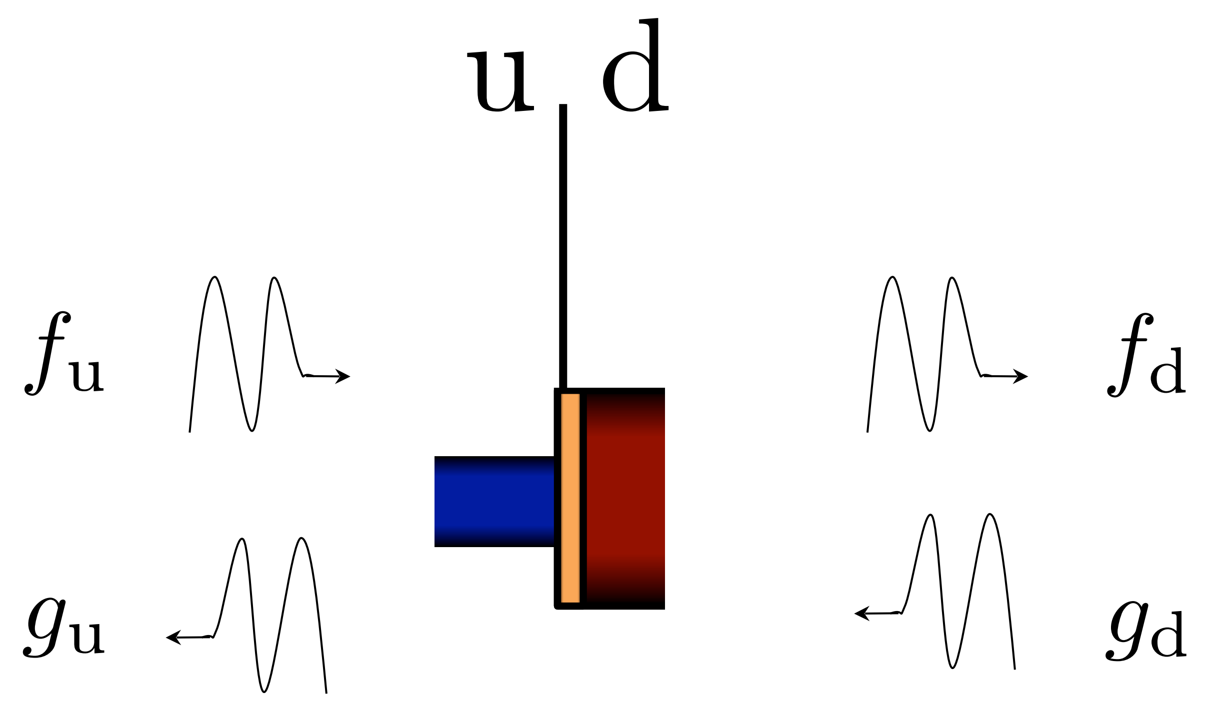

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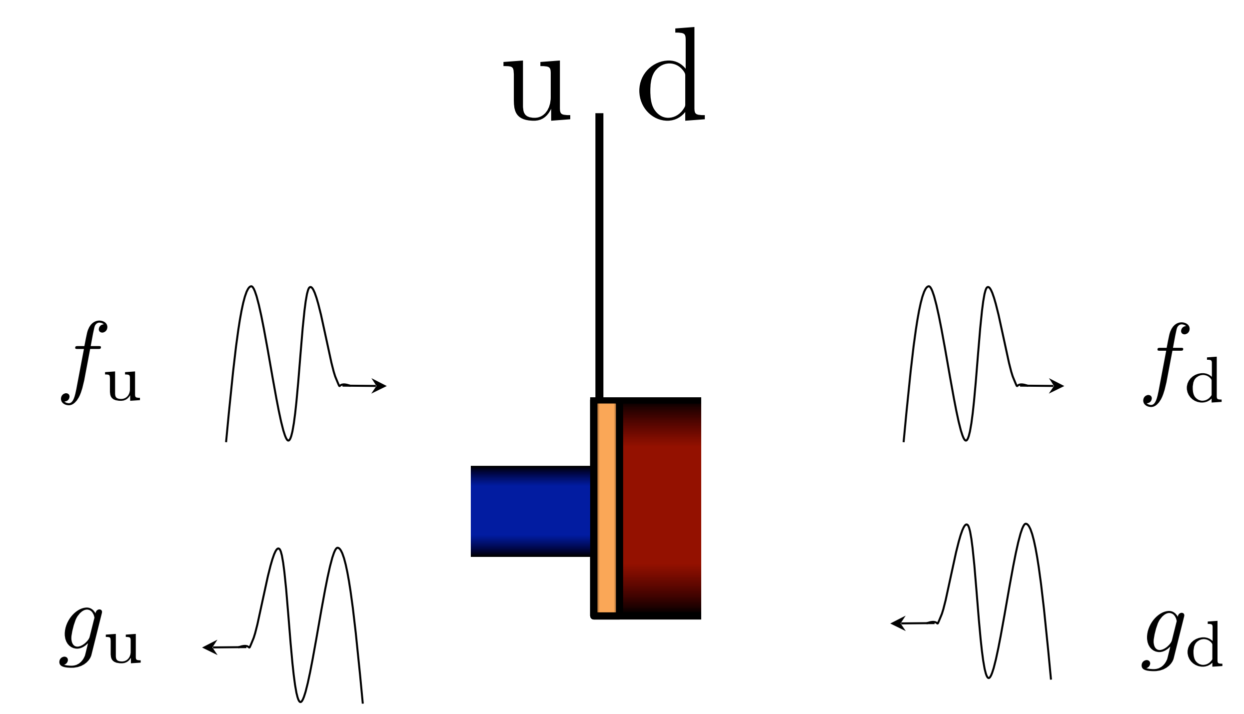

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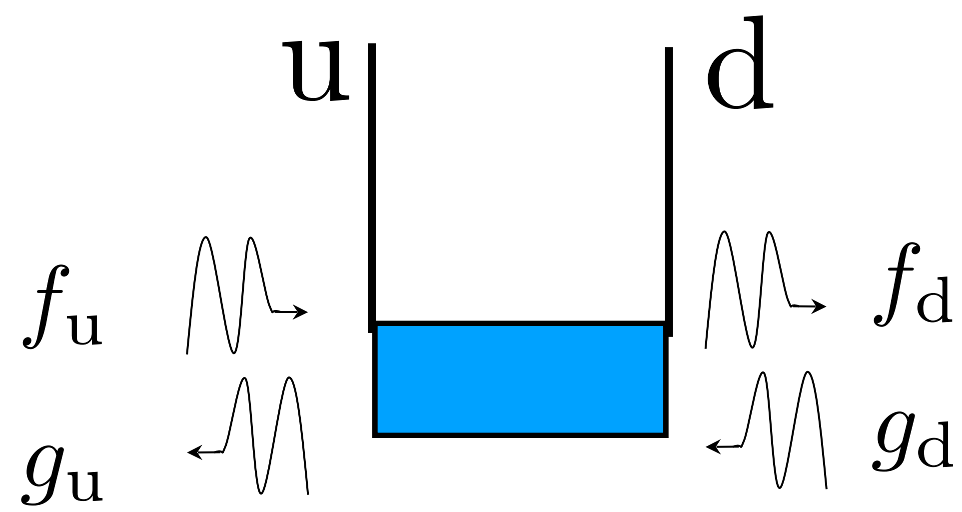
$$\begin{bmatrix} f_\mathrm{d} \\ g_\mathrm{d} \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} T_{F,11}(s) & T_{F,12}(s) \\ T_{F,21}(s) & T_{F,22}(s) \end{bmatrix}}_{T_\mathrm{F}(s)} \begin{bmatrix} f_\mathrm{u} \\ g_\mathrm{u} \end{bmatrix}$$

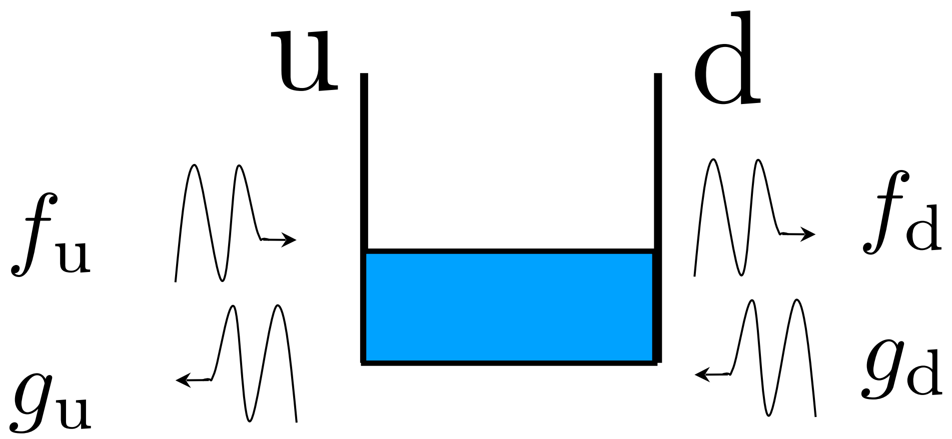
Each coefficient may be a transcendental function in s

The relation of upstream and downstream waves in a duct is straightforward



$$\begin{bmatrix} f_\mathrm{d} \\ g_\mathrm{d} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-sl/\bar{c}} & 0 \\ 0 & e^{sl/\bar{c}} \end{bmatrix}}_{T_D(s)} \begin{bmatrix} f_\mathrm{u} \\ g_\mathrm{u} \end{bmatrix}$$

All elements of the system can be concatenated to form an unique matrix



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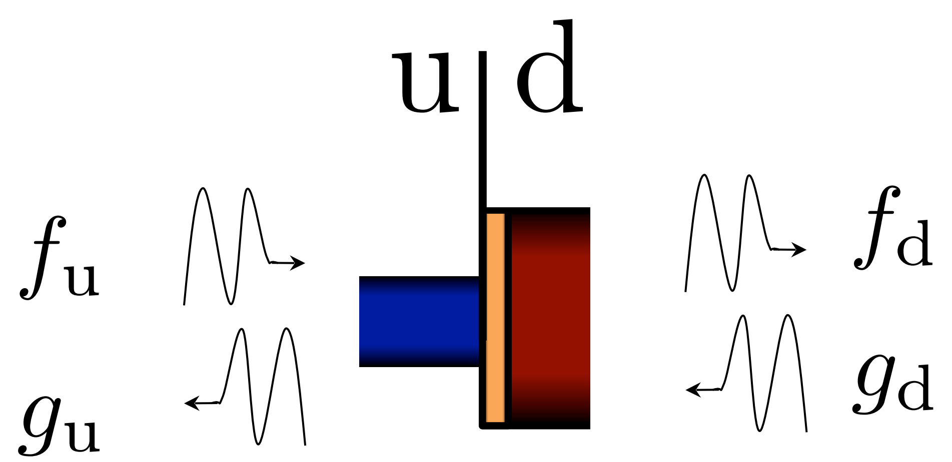

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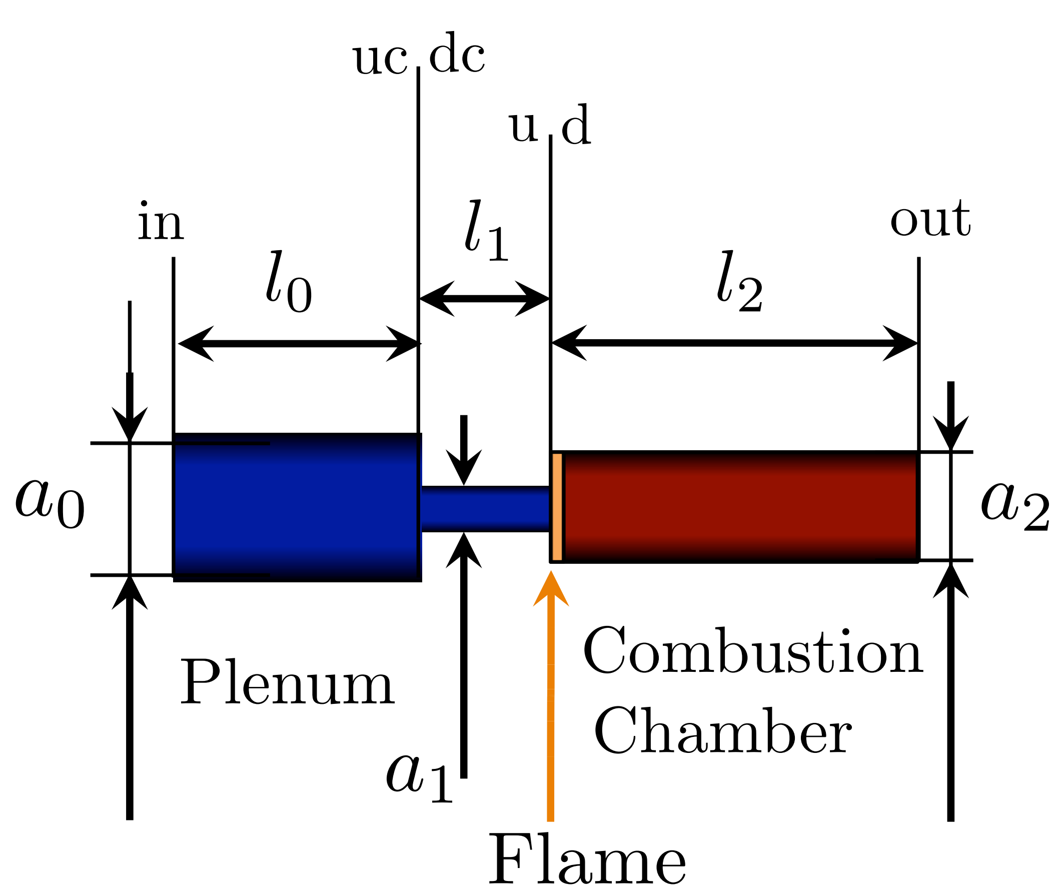

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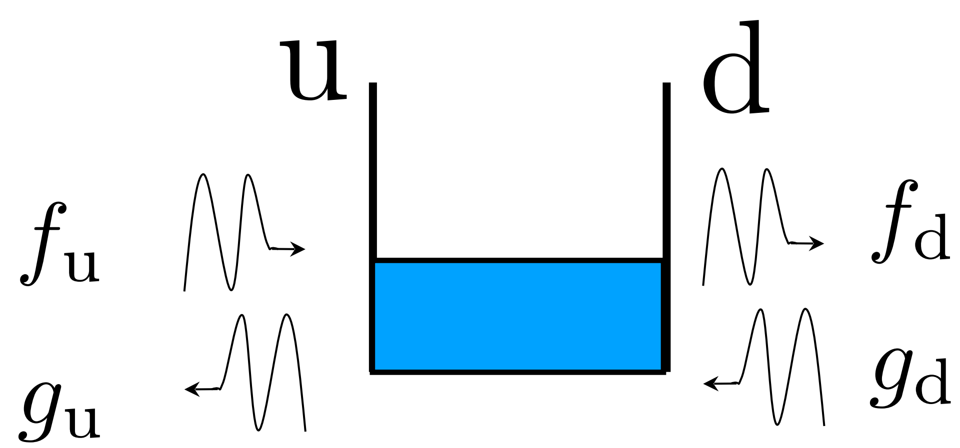

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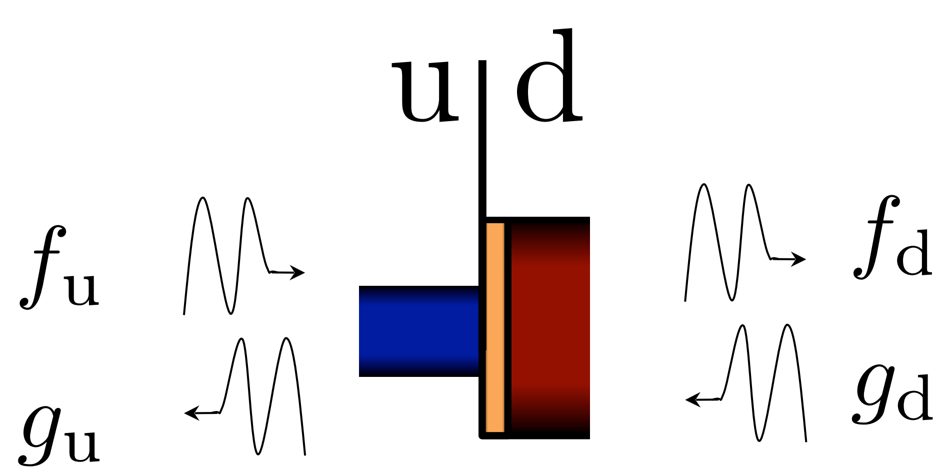

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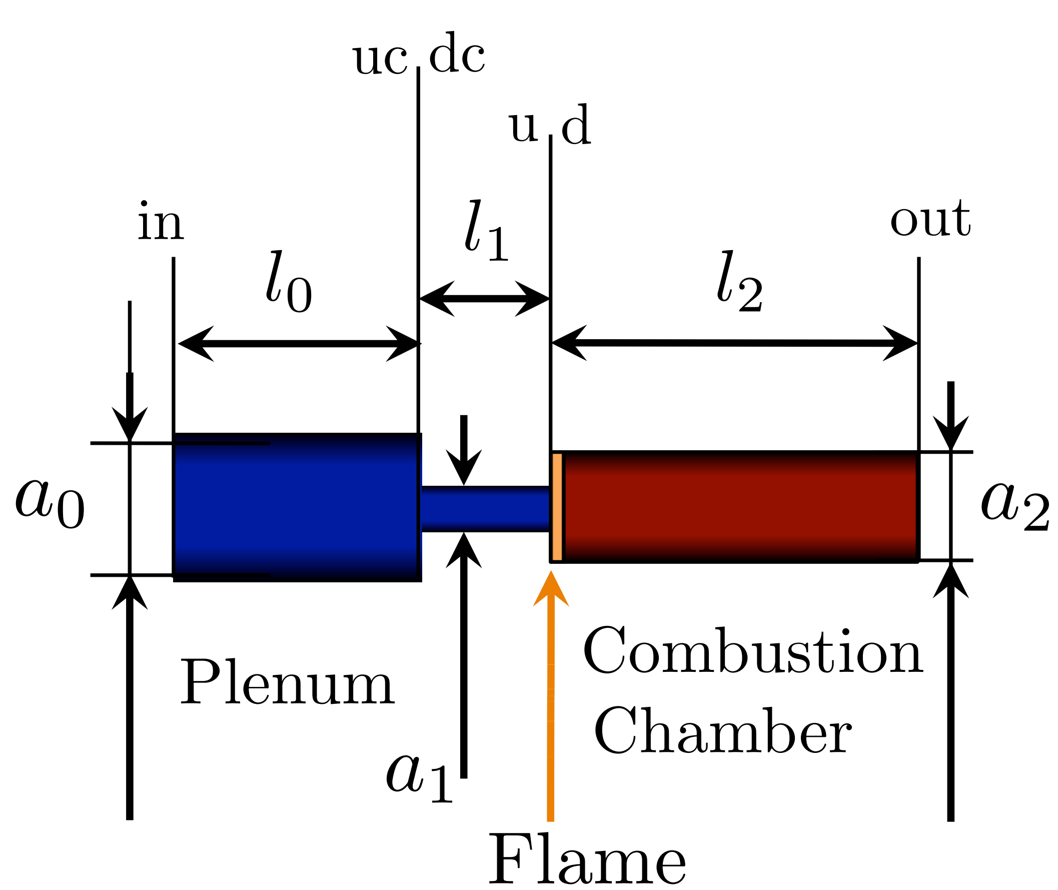

$$T(s) = T_{D2} T_F T_{D1} T_C T_{D0}$$

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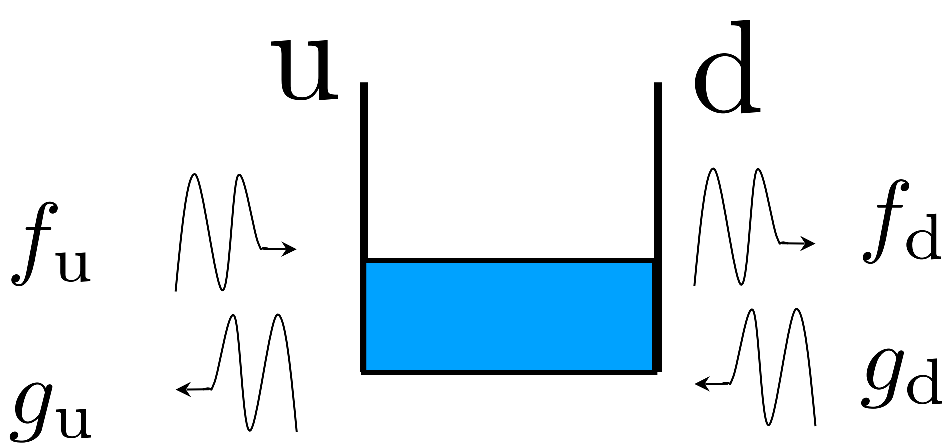

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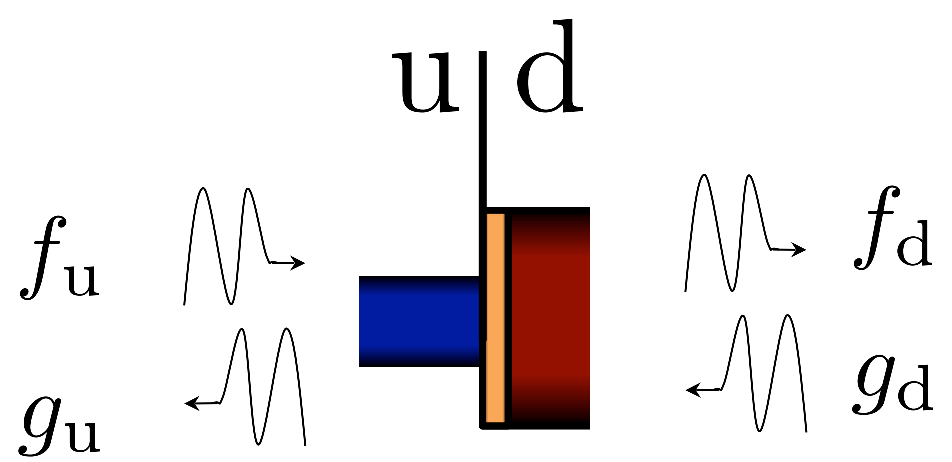

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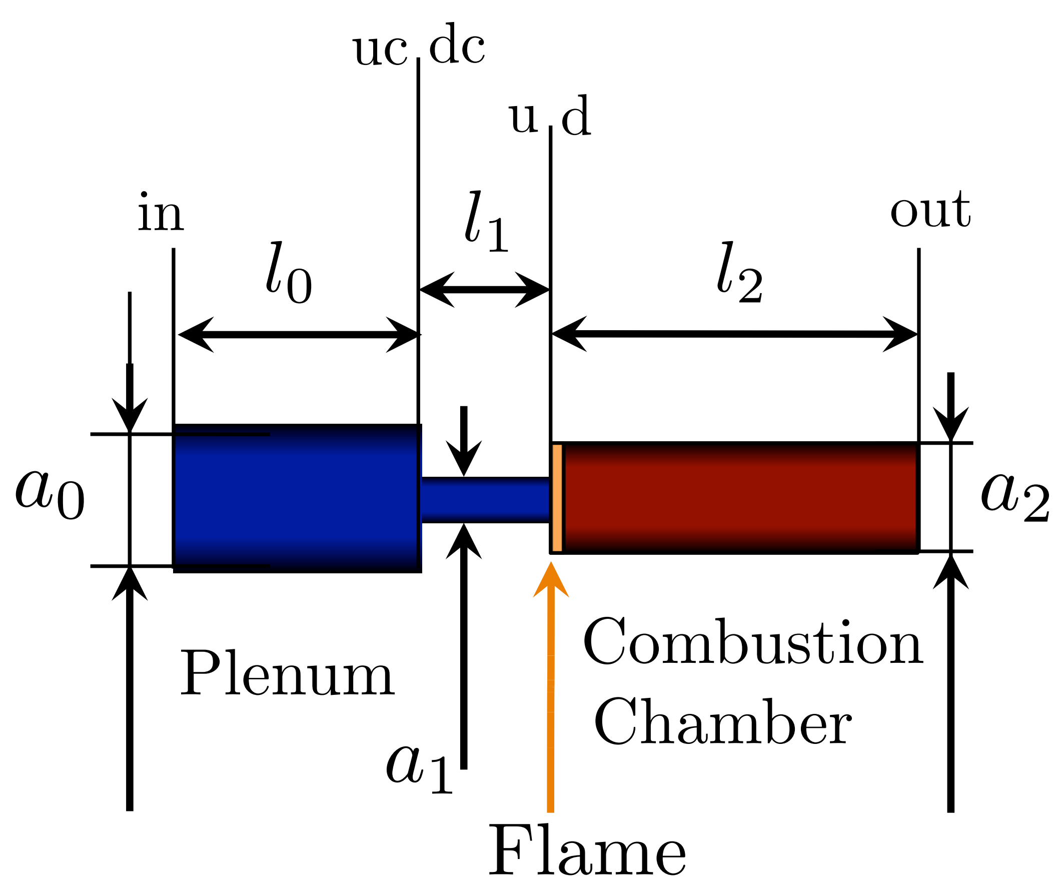

$$\begin{bmatrix} f_{\mathrm{out}} \\ g_{\mathrm{out}} \end{bmatrix} = \underbrace{\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix}}_{T(s)} \begin{bmatrix} f_{\mathrm{in}} \\ g_{\mathrm{in}} \end{bmatrix}.$$

where

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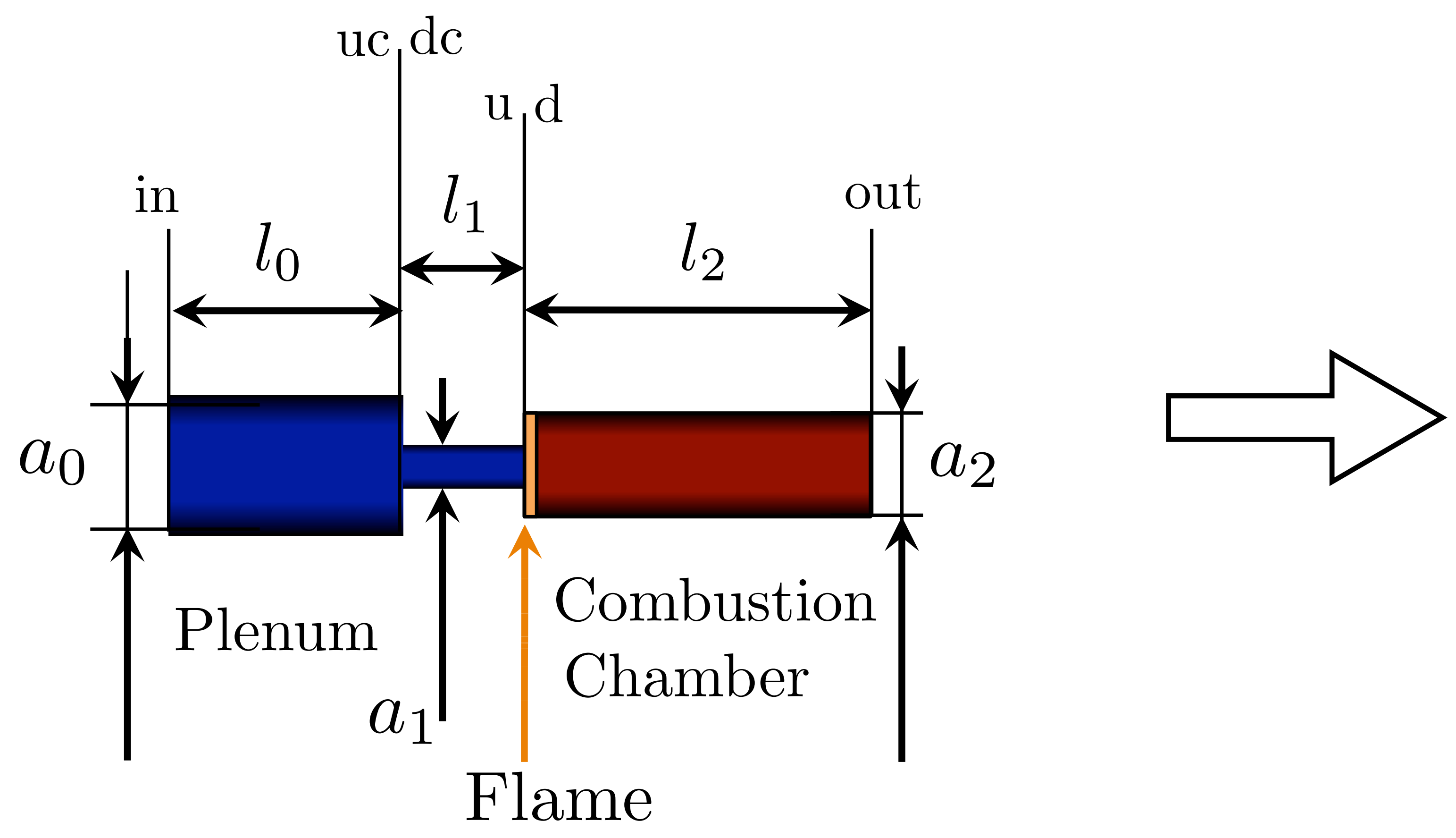

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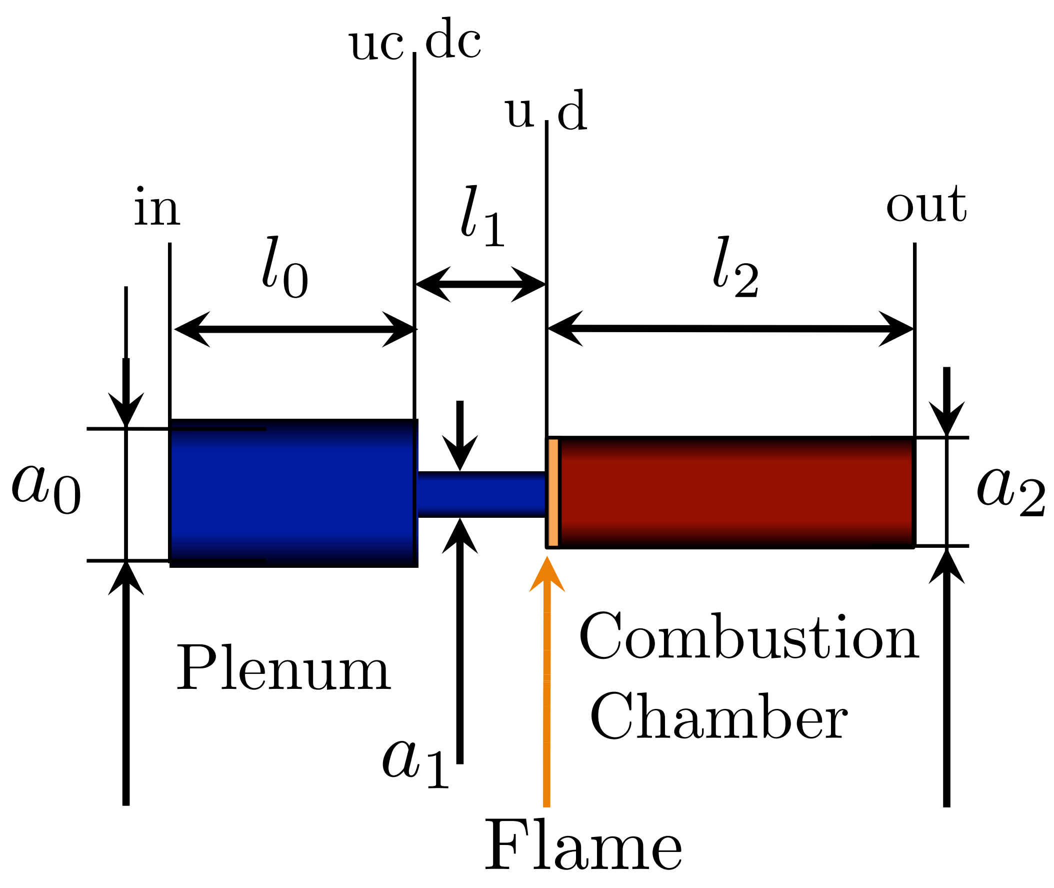

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with known reflection coefficients

$$R_{\mathrm{in}} = \frac{f_{\mathrm{in}}}{g_{\mathrm{in}}} \qquad R_{\mathrm{out}} = \frac{g_{\mathrm{out}}}{f_{\mathrm{out}}}$$

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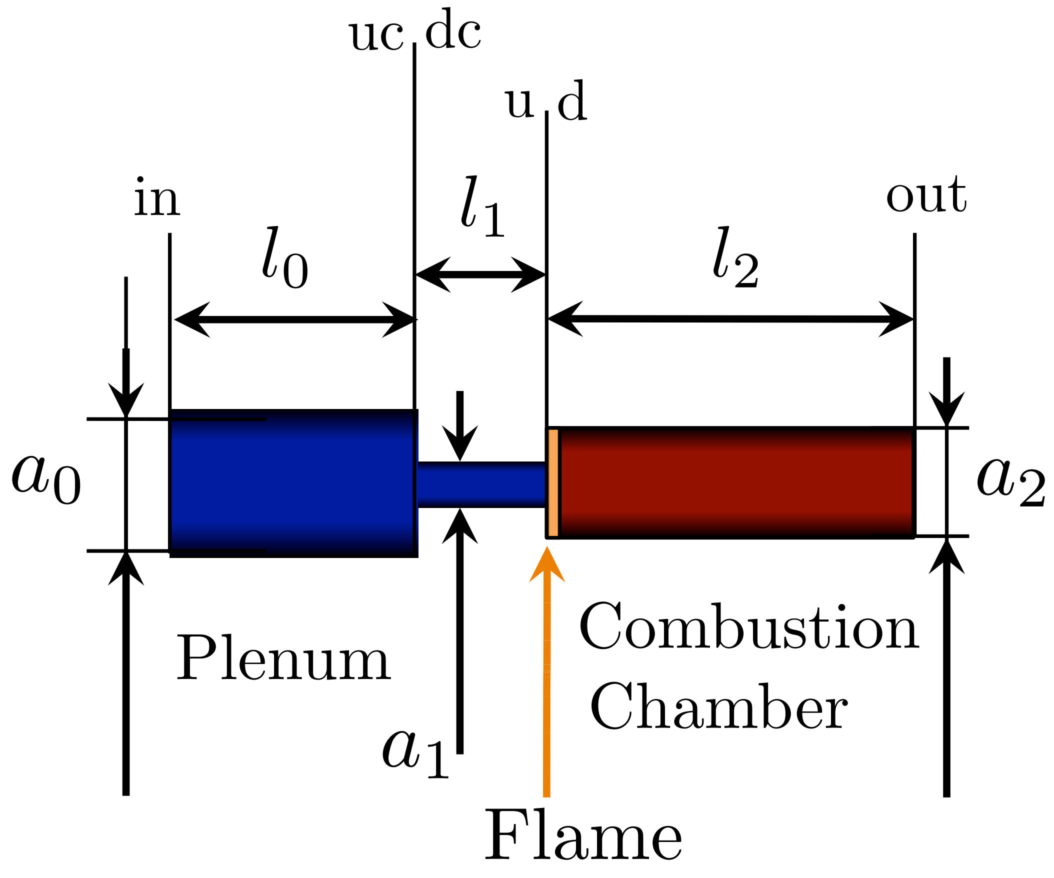

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$$\underbrace{\begin{bmatrix} 1 & -R_{\text{in}} & 0 & 0 \\ 0 & 0 & -R_{\text{out}} & 1 \\ T_{11}(s) & T_{12}(s) & -1 & 0 \\ T_{21}(s) & T_{22}(s) & 0 & -1 \end{bmatrix}}_{M(s)} \begin{bmatrix} f_{\text{in}} \\ g_{\text{in}} \\ f_{\text{out}} \\ g_{\text{out}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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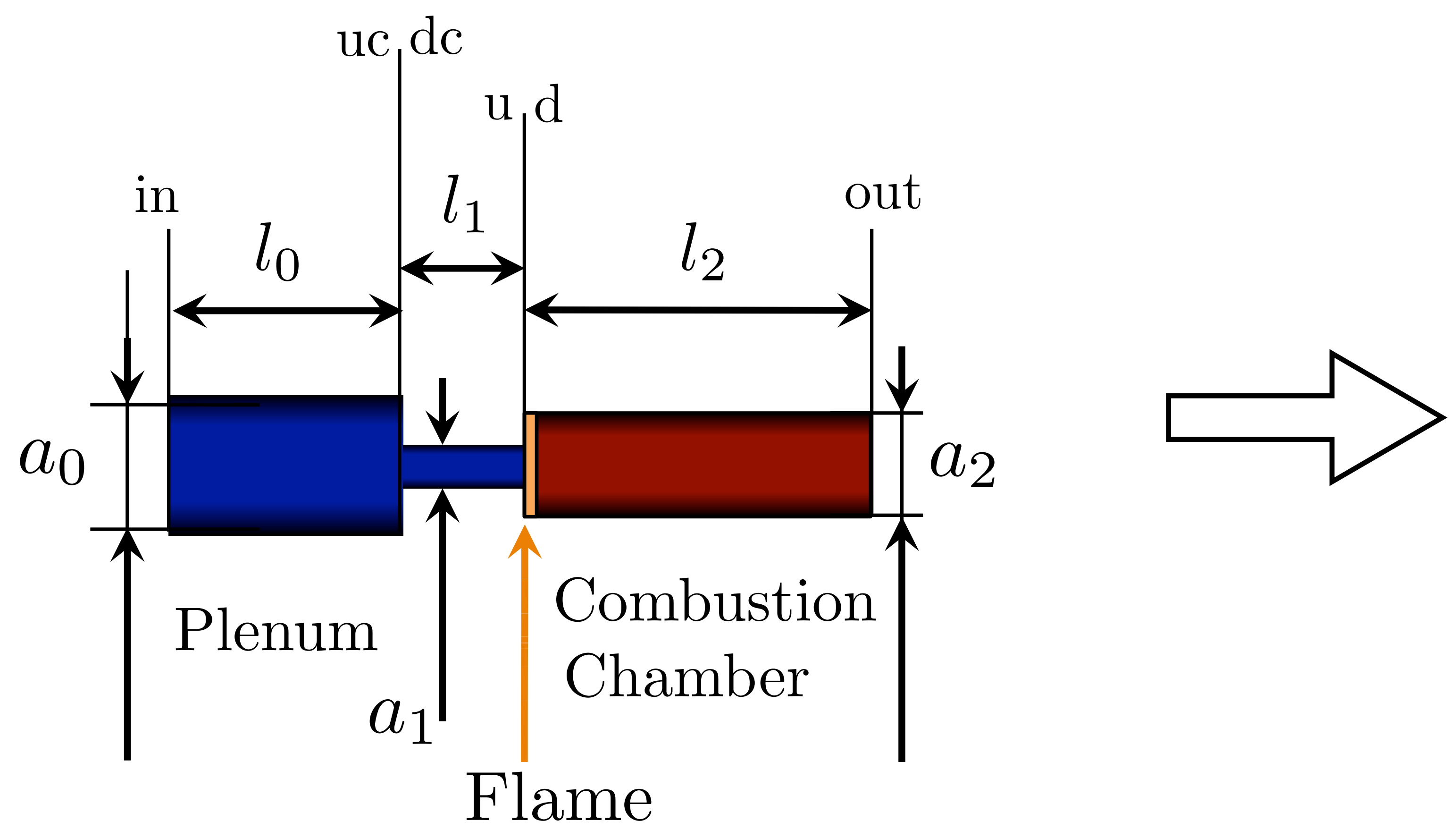

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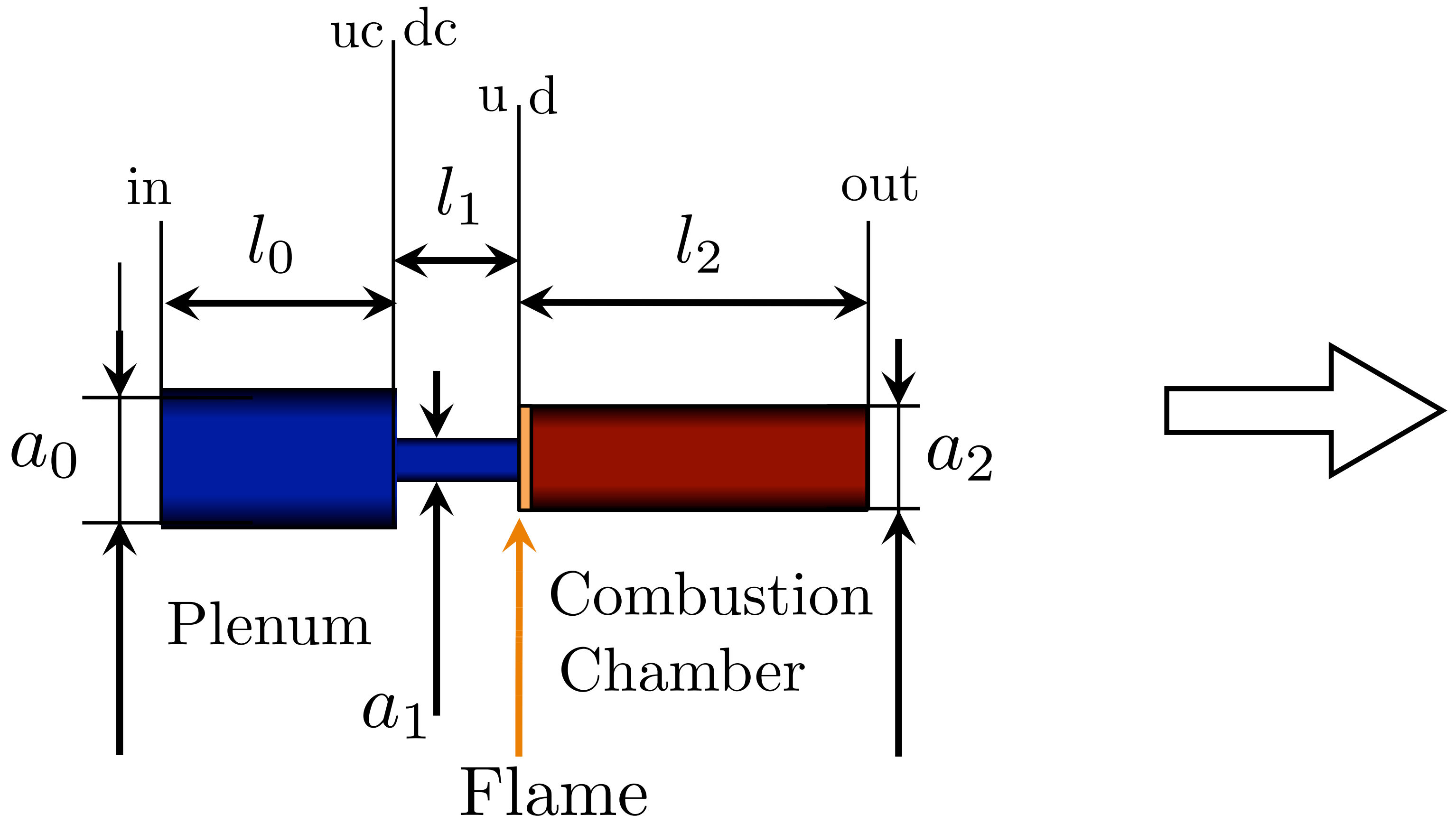
The eigenvalues of the system are obtained by doing $\det(M(s)) = 0$, which results in

$$\boxed{T_{22}(s) - R_{\text{out}} T_{12}(s) + R_{\text{in}} T_{21}(s) - R_{\text{in}} R_{\text{out}} T_{11}(s) = 0}$$

All elements of the system can be concatenated to form an unique matrix



$$\begin{bmatrix} f_{\text{out}} \\ g_{\text{out}} \end{bmatrix} = \underbrace{\begin{bmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{bmatrix}}_{T(s)} \begin{bmatrix} f_{\text{in}} \\ g_{\text{in}} \end{bmatrix}.$$

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For today afternoon

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$$T_{22}(s) - R_{\text{out}} T_{12}(s) + R_{\text{in}} T_{21}(s) - R_{\text{in}} R_{\text{out}} T_{11}(s) = 0$$

All approaches so far result in a nonlinear eigenvalue problem

Linearized Navier Stokes Equations

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho} u_j' + \rho' \bar{u}_j\right) = 0$$

$$\frac{\partial}{\partial t}\left(\bar{\rho} u_i' + \rho' \bar{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_i u_j' + \bar{\rho} u_i' \bar{u}_j + \rho' \bar{u}_i \bar{u}_j\right) = -\frac{\partial p'}{\partial x_i} + \frac{\partial \tau_{ij}'}{\partial x_j}$$

$$\bar{T}\left[\frac{\partial}{\partial t}\left(\bar{\rho} s' + \rho' \bar{s}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j s' + \bar{\rho} u_j' \bar{s} + \rho' \bar{u}_j \bar{s}\right)\right] + T' \frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_j \bar{s}\right) = \dot{q}'$$

Helmholtz Equation

$$s^2 \hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}$$

Network model

$$\det(M(s)) = 0$$

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Linearized Navier Stokes Equations

Helmholtz E

Network mo

Nonlinear eigenvalue problems present many difficulties:

- Iterative approaches are needed. They may not always converge

All approaches so far result in a nonlinear eigenvalue problem

Linearized Navier Stokes Equations

Helmholtz E

Network mo

Nonlinear eigenvalue problems present many difficulties:

- Iterative approaches are needed. They may not always converge.
- Usually, only one eigenvalue can be computed at a time

All approaches so far result in a nonlinear eigenvalue problem

Linearized Navier Stokes Equations

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Nonlinear eigenvalue problems present many difficulties:

- Iterative approaches are needed. They may not always converge.
- Usually, only one eigenvalue can be computed at a time
- There are eigenvalues that, due to their small basin of attraction (associated with the iterative solver) cannot be captured

All approaches so far result in a nonlinear eigenvalue problem

Linearized Navier Stokes Equations

Can we do better?

Helmholtz E

Network mo

All approaches so far result in a nonlinear eigenvalue problem

Linearized Navier Stokes Equations

Can we do better?

Yes! By writing the obtained system of equations under a state space formalism

Helmholtz E

Network mo

Outline

- From the Navier-Stokes equations to the acoustic jump conditions
- From primitive variables to acoustic invariants (waves)
- The state space approach

Each subsystem of the system can be expressed as a state-space model

state variable

rate of change of state variable

input

$$\dot{\boldsymbol{x}} = \tilde{A}\boldsymbol{x} + \tilde{B}\tilde{\boldsymbol{u}}$$

output

$$\boldsymbol{y} = \tilde{C}\boldsymbol{x} + \tilde{D}\tilde{\boldsymbol{u}}$$

$$\tilde{\boldsymbol{u}} = F\boldsymbol{y} + \boldsymbol{u}$$

Each subsystem of the system can be expressed as a state-space model

system matrix

$$\dot{\boldsymbol{x}} = \tilde{A}\boldsymbol{x} + \tilde{B}\tilde{\boldsymbol{u}}$$

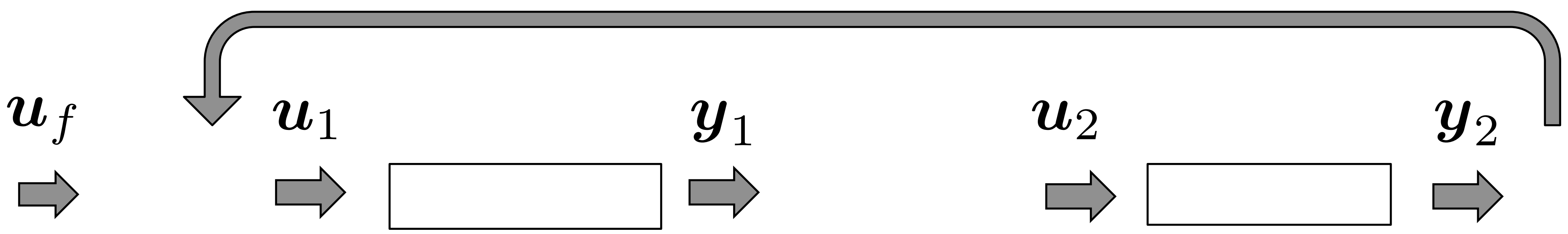
feedthrough matrix

$$\boldsymbol{y} = \tilde{C}\boldsymbol{x} + \tilde{D}\tilde{\boldsymbol{u}}$$

$$\tilde{\boldsymbol{u}} = F\boldsymbol{y} + \boldsymbol{u}$$

feedback matrix

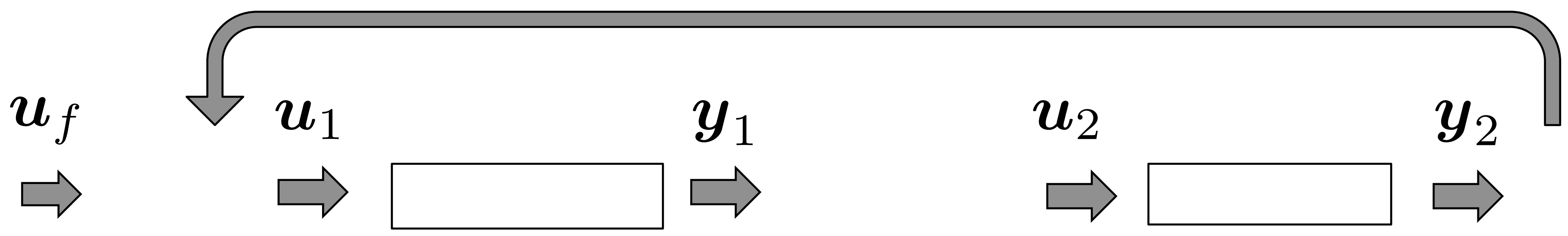
Each subsystem of the system can be expressed as a state-space model



$$\begin{aligned} \dot{\boldsymbol{x}}_1 &= A_1 \boldsymbol{x}_1 + B_1 \boldsymbol{u}_1 \\ \boldsymbol{y}_1 &= C_1 \boldsymbol{x}_1 + D_1 \boldsymbol{u}_1 \end{aligned}$$

$$\begin{aligned} \dot{\boldsymbol{x}}_2 &= A_2 \boldsymbol{x}_2 + B_2 \boldsymbol{u}_2 \\ \boldsymbol{y}_2 &= C_2 \boldsymbol{x}_2 + D_2 \boldsymbol{u}_2 \end{aligned}$$

Each subsystem of the system can be expressed as a state-space model



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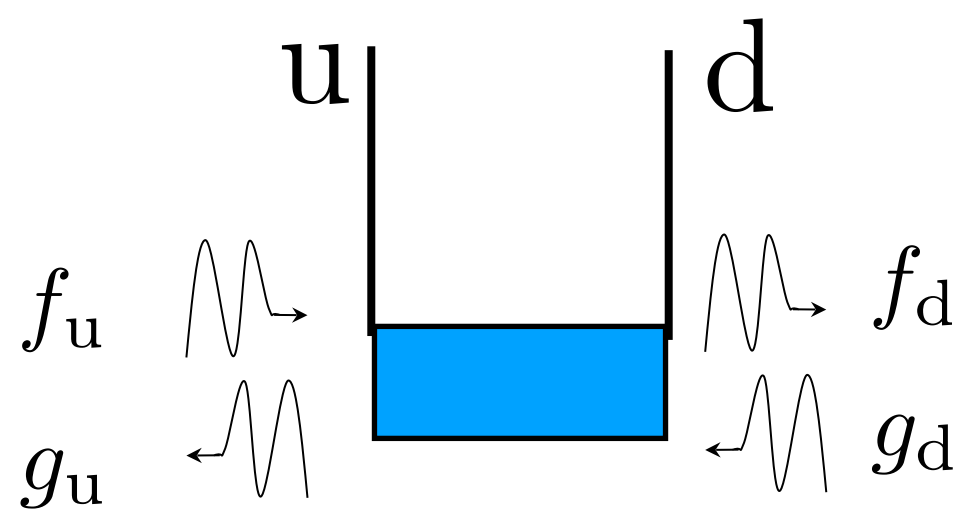
$$\underbrace{\begin{bmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{bmatrix}}_{\tilde{\boldsymbol{u}}} = \underbrace{\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \end{bmatrix}}_{\boldsymbol{y}} + \underbrace{\begin{bmatrix} \boldsymbol{u}_f \\ \end{bmatrix}}_{\boldsymbol{u}}$$

We should think in ‘time’ for a representation in state-space

Example: acoustic propagation in a duct



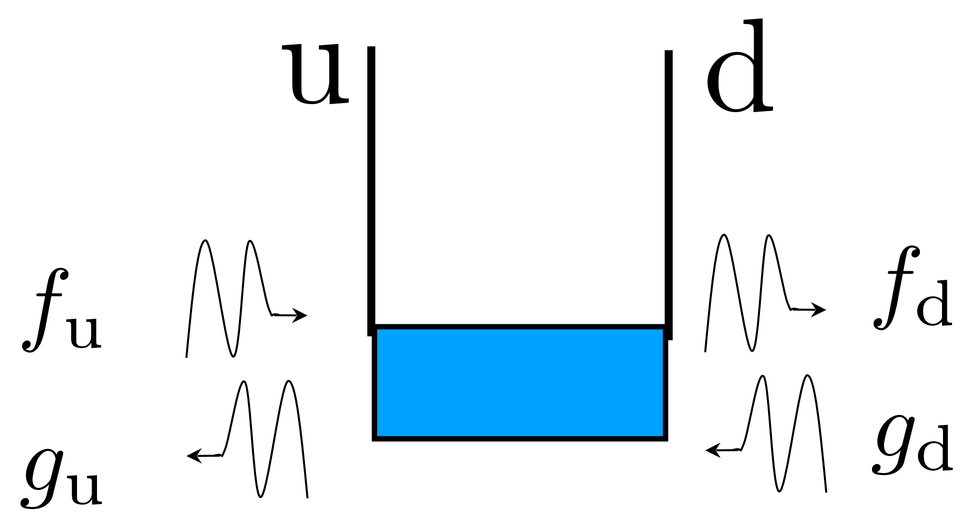
Frequency domain representation

$$\begin{bmatrix} f_{\mathrm{d}} \\ g_{\mathrm{d}} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{-sl/\bar{c}} & 0 \\ 0 & e^{sl/\bar{c}} \end{bmatrix}}_{T_D(s)} \begin{bmatrix} f_{\mathrm{u}} \\ g_{\mathrm{u}} \end{bmatrix}$$

Time domain representation

We should think in ‘time’ for a representation in state-space

Example: acoustic propagation in a duct



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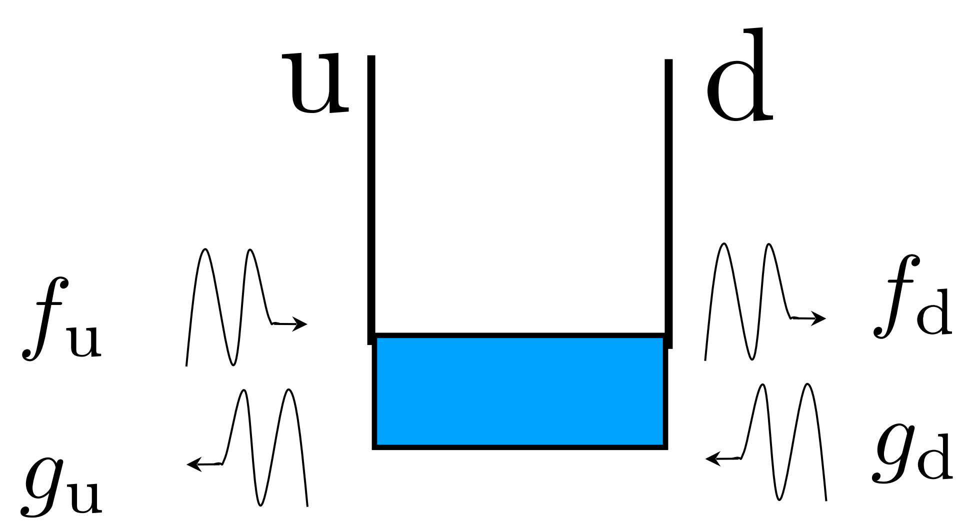
Time domain representation

$$\dot{f} + \bar{c}\frac{\partial f}{\partial x} = 0; \quad \dot{g} - \bar{c}\frac{\partial g}{\partial x} = 0$$

discretized by a numerical scheme

We should think in ‘time’ for a representation in state-space

Example: acoustic propagation in a duct



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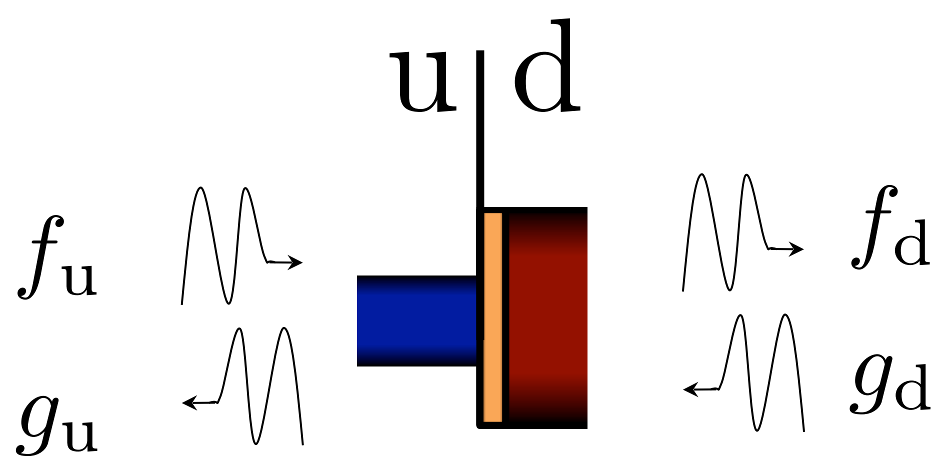
Time domain representation

$$\dot{f} + \bar{c}\frac{\partial f}{\partial x} = 0; \quad \dot{g} - \bar{c}\frac{\partial g}{\partial x} = 0 \quad \Longrightarrow \quad \begin{bmatrix} \dot{\boldsymbol{f}} \\ \dot{\boldsymbol{g}} \end{bmatrix} = \begin{bmatrix} A_\mathrm{d} & \\ & A_\mathrm{u} \end{bmatrix} \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}$$

discretized by a numerical scheme

We should think in ‘time’ for a representation in state-space

Example: Cross section Jump with Flame



Frequency domain representation

$$\begin{bmatrix} f_\mathrm{d} \\ g_\mathrm{d} \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} T_{F,11}(s) & T_{F,12}(s) \\ T_{F,21}(s) & T_{F,22}(s) \end{bmatrix}}_{T_\mathrm{F}(s)} \begin{bmatrix} f_\mathrm{u} \\ g_\mathrm{u} \end{bmatrix}$$

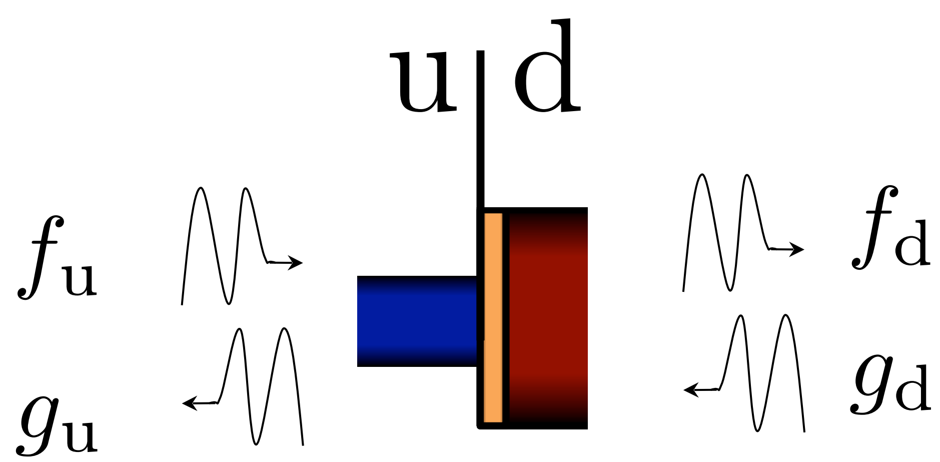
⇓

The nonlinearity in here is due to the flame response

$$\dot{Q}' = G(s) e^{\phi(s)} \; u$$

We should think in 'time' for a representation in state-space

Example: Cross section Jump with Flame



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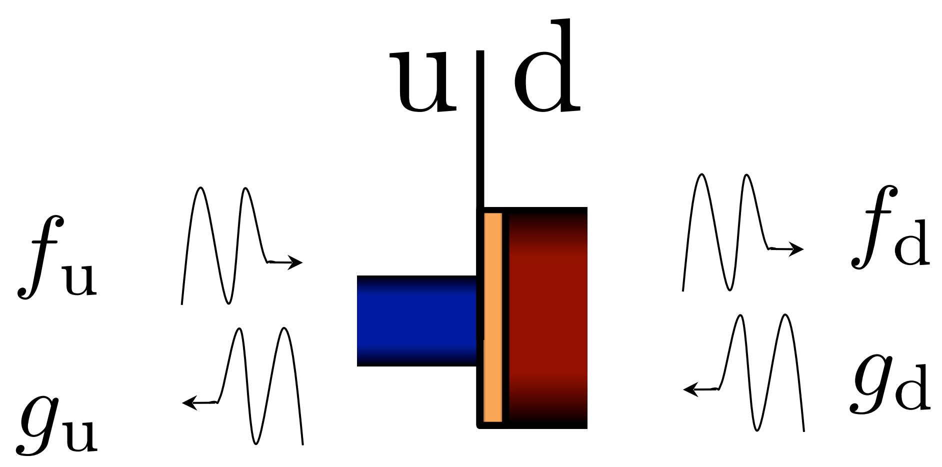
How to think in time?

We should think in ‘time’ for a representation in state-space

Example: Cross section Jump with Flame



The nonlinearity in here is due to the flame response

$$\dot{Q}' = G(s)e^{\phi(s)}\ u$$

How to think in time?

Possibility 1

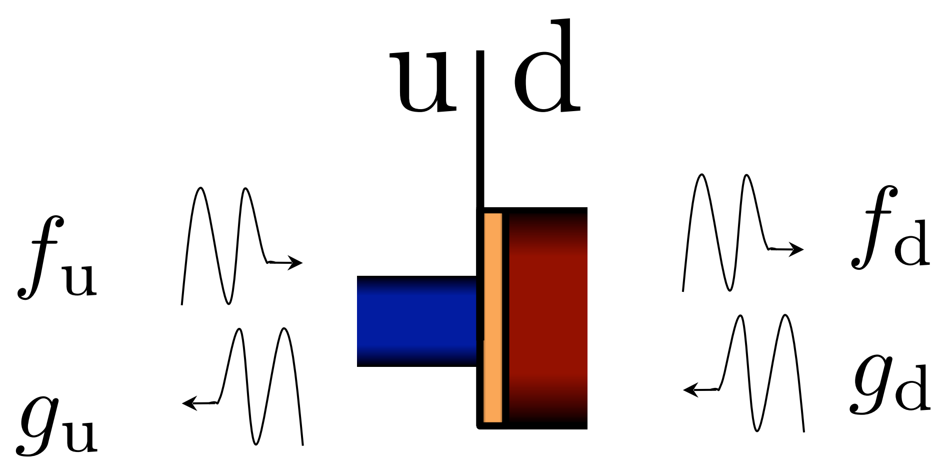
$$\dot{\boldsymbol{x}} = \tilde{A}\boldsymbol{x} + \tilde{B}\tilde{\boldsymbol{u}}$$

$$\dot{Q}' = \tilde{C}\boldsymbol{x} + \tilde{D}\tilde{\boldsymbol{u}}$$

Figure out a model (set of ODEs) that mimic the behavior

We should think in ‘time’ for a representation in state-space

Example: Cross section Jump with Flame



The nonlinearity in here is due to the flame response

$$\dot{Q}' = G(s)e^{\phi(s)}\ u$$

How to think in time?

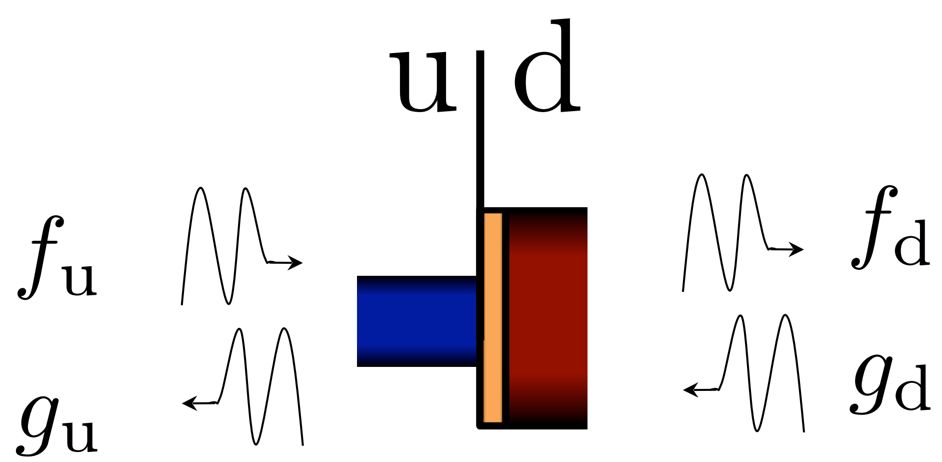
Possibility 2

Think in discrete time and a convolution Equation

The output is a ‘filtered’ input

We should think in ‘time’ for a representation in state-space

Example: Cross section Jump with Flame



The nonlinearity in here is due to the flame response

$$\dot{Q}' = G(s)e^{\phi(s)}\ u$$

How to think in time?

Possibility 2

unit impulse response

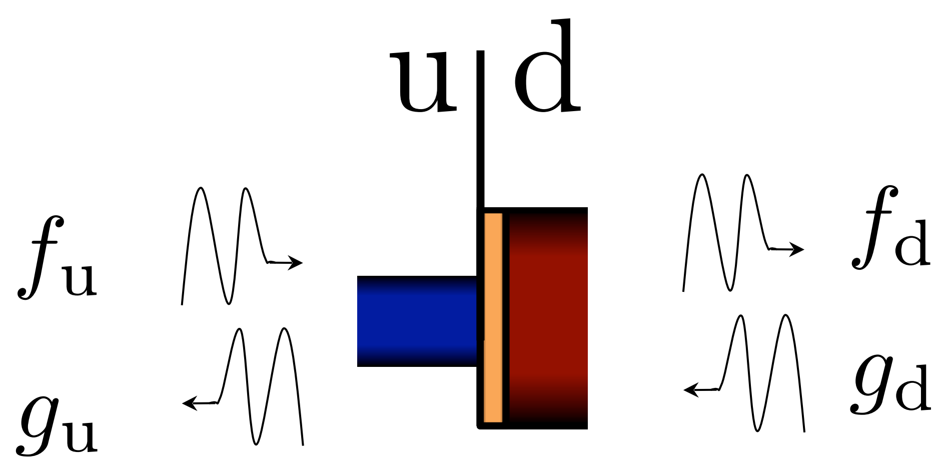
$$\dot{Q}'_n = \sum_{k=0}^{L} h_k u_{n-k}$$

Think in discrete time and a convolution Equation

The output is a ‘filtered’ input

We should think in ‘time’ for a representation in state-space

Example: Cross section Jump with Flame



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How to think in time?

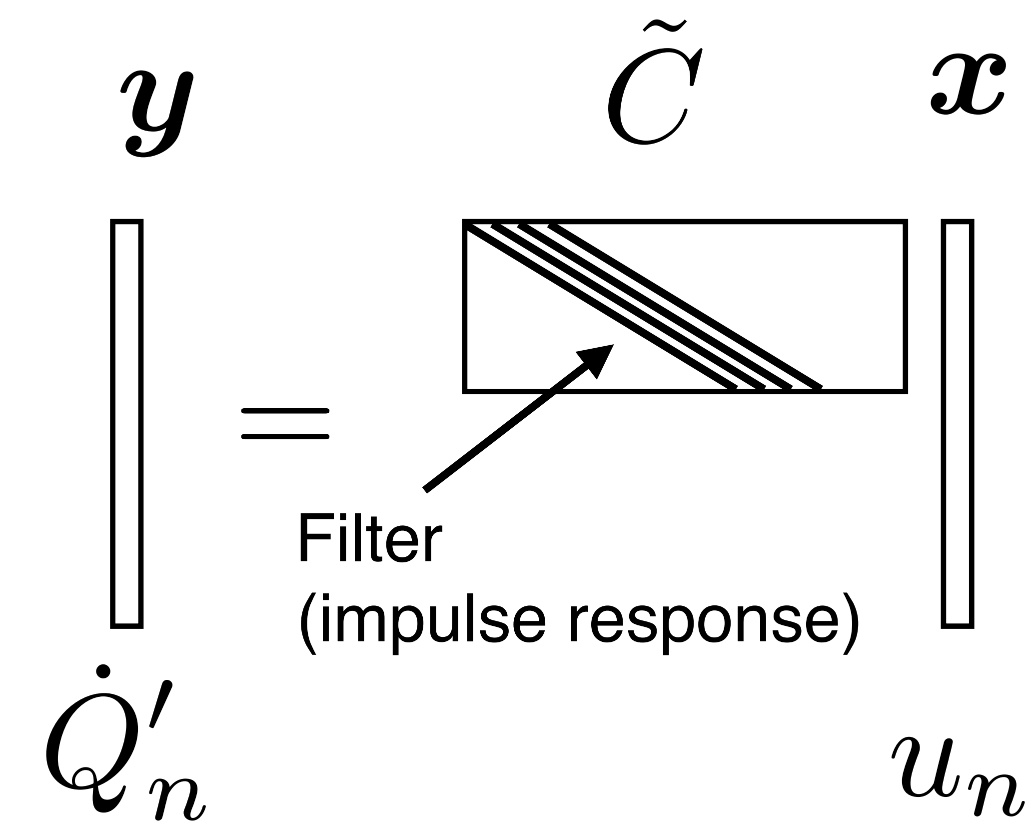
Possibility 2

$$\dot{Q}'_n = \sum_{k=0}^{L} h_k u_{n-k}$$

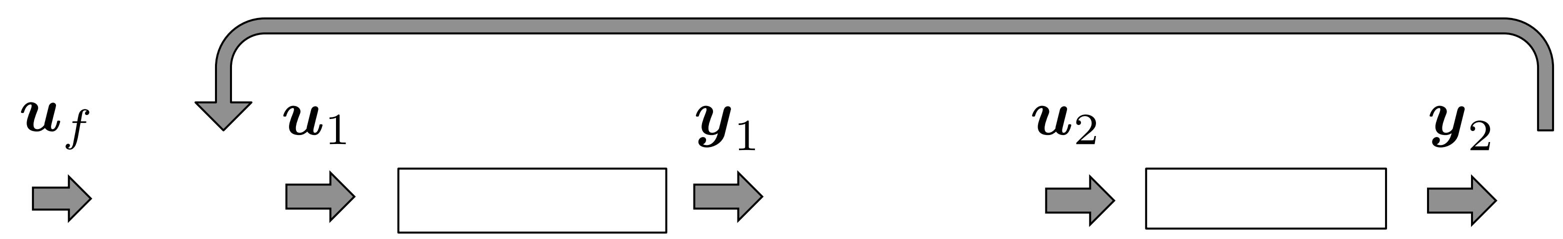
Matrix representing convection

$$\Delta \boldsymbol{x} = \tilde{A}\boldsymbol{x} + \tilde{B}\boldsymbol{u}$$

$$\boldsymbol{y} = \tilde{C}\boldsymbol{x} + \tilde{D}\tilde{\boldsymbol{u}}$$



Each subsystem of the system can be expressed as a state-space model



$$\dot{\boldsymbol{x}}_1 = A_1 \boldsymbol{x}_1 + B_1 \boldsymbol{u}_1$$
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$$\tilde{A} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}$$
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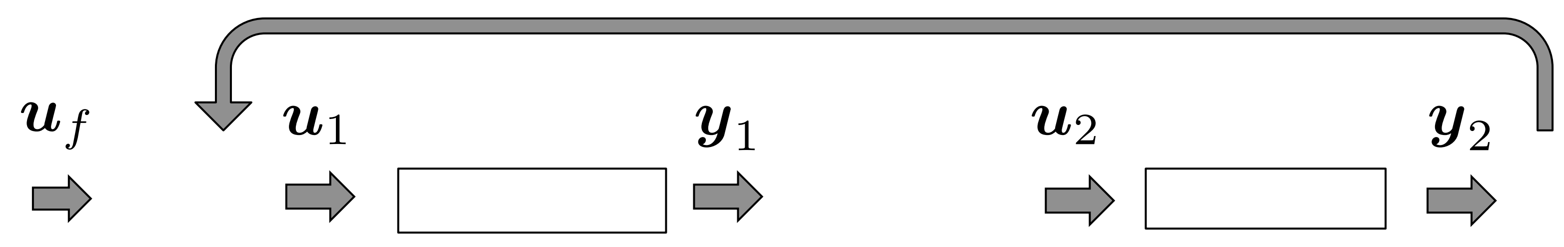

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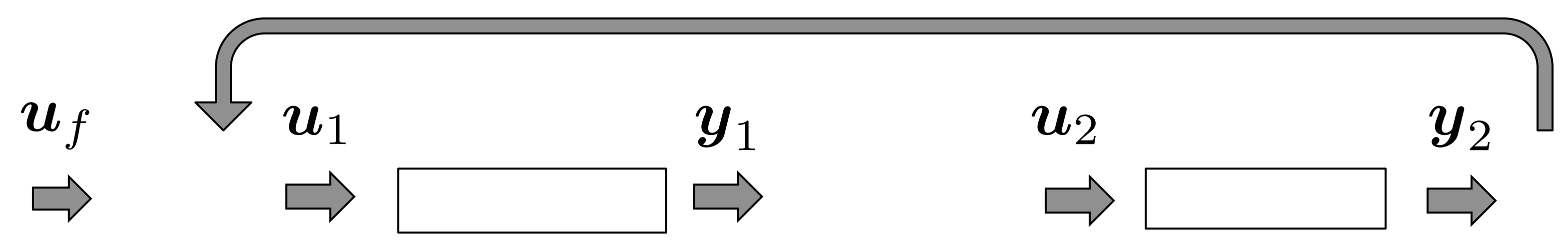
A global expression can be obtained by combining the equations

$$\dot{\boldsymbol{x}} = \underbrace{\left(\tilde{A} + \tilde{B}(I - F\tilde{D})^{-1}F\tilde{C}\right)}_{A}\boldsymbol{x} + \underbrace{\tilde{B}(I - F\tilde{D})^{-1}\boldsymbol{u}}_{-\boldsymbol{b}}$$

Global system matrix

Each subsystem of the system can be expressed as a state-space model



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Global system matrix

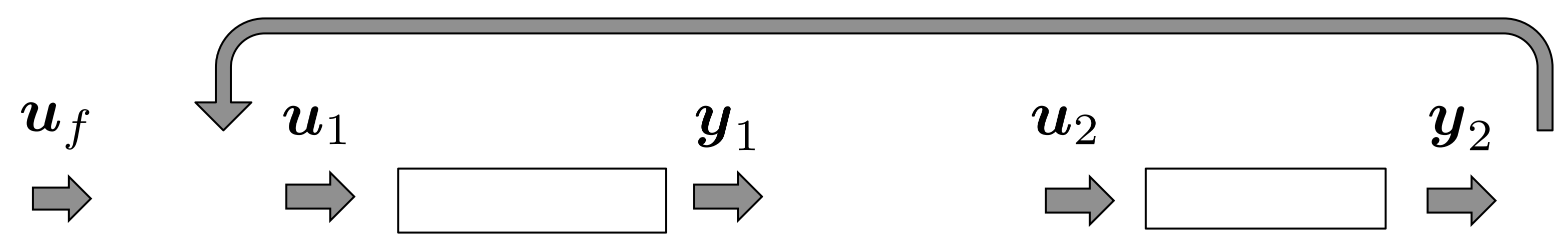
Time domain

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} - \boldsymbol{b}$$

Frequency domain

$$A\hat{\boldsymbol{x}} - s\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}} \qquad A\hat{\boldsymbol{x}} = s\hat{\boldsymbol{x}}$$

The system matrix A is constant!



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Global system matrix

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Frequency domain

$$A\hat{\boldsymbol{x}} - s\hat{\boldsymbol{x}} = \hat{\boldsymbol{b}}$$

$$A\hat{\boldsymbol{x}} = s\hat{\boldsymbol{x}}$$

The system matrix A is constant! A linear eigenvalue problem can be obtained

Linear Eigenvalue Problem

$$A\hat{\boldsymbol{x}} = s\hat{\boldsymbol{x}}$$

Advantages:

- There are many efficient algorithms to solve a linear eigenvalue problem
- It is possible to obtain without many difficulties ALL the eigenvalues of the system in one shot!
- Finding or not finding an eigenvalue is not an issue anymore as iterative solvers (and corresponding basin of attraction) do not apply

How to model systems that are complex in frameworks that are easy for computation?

Let us take a look at reduced order models