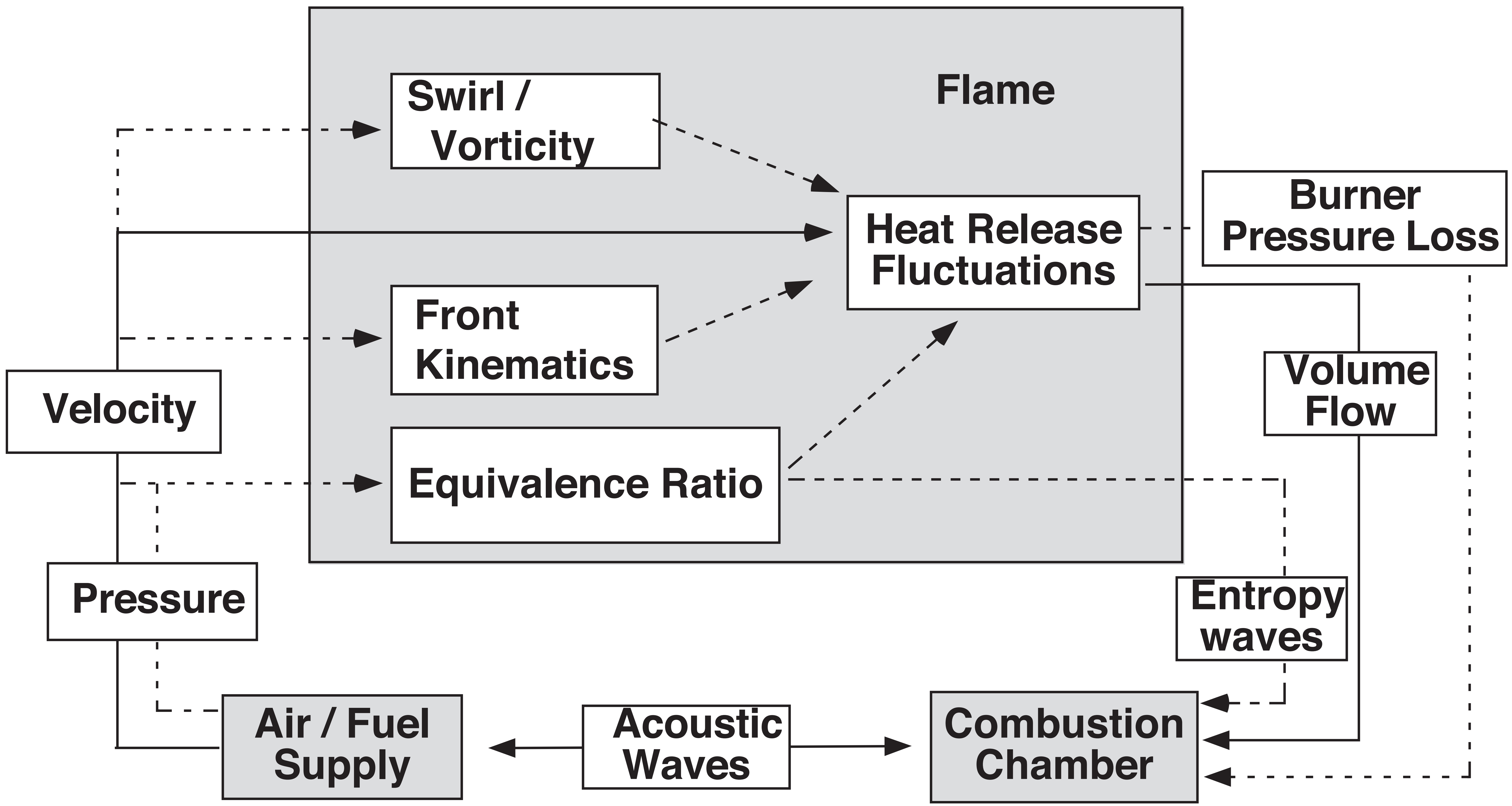
About the Wave Equation for reacting flows

Camilo F. Silva

April 27, 2022

Can we model this?



Sattelmayer (1997)

Let us start with the equations for compressible reactive flows!

Outline

- $\boxed{\text{From the Navier-Stokes equations to the LRF and LNSE}}$

- From the Navier-Stokes equations to the wave equation

- Recapitulating: What is the Helmholtz Equation good for?

Nothing but conservation of mass, momentum, energy and species

mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0 \qquad \Rightarrow \qquad \frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{\partial u_j}{\partial x_j}$$

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$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \qquad \Rightarrow \qquad \rho \frac{D u_i}{Dt} = -\frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

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energy

$$\frac{\partial \rho s}{\partial t} + \frac{\partial \rho u_j s}{\partial x_j} = \frac{\dot{q}}{T} \quad \Rightarrow \quad \rho T\frac{Ds}{Dt} = \dot{q}$$

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species

$$\frac{\partial \rho Y_k}{\partial t} + \frac{\partial \rho u_j Y_k}{\partial x_j} = \frac{\partial}{\partial x_j}\left(D_k \frac{\partial Y_k}{\partial x_j}\right) + \dot{\Omega}_k \quad \Rightarrow \quad \rho \frac{DY_k}{Dt} = \frac{\partial}{\partial x_j}\left(D_k \frac{\partial Y_k}{\partial x_j}\right) + \dot{\Omega}_k$$

Now linearization of equations is performed. Why? because we want to evaluate instability

We want a system that can be written as $Ax = 0$

Therefore we do $[] = \bar{[]} + []'$

Meindl et al. 2021

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This equations are known as the **Linearized Reactive Flow (LRF)** equations

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The LRF equations still have to probe their utility in turbulent flows

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The LNSE equations are simplified LRF equations

This equations are known as the **Linearized Navier Stokes (LNSE)** equations

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Requires a flame response

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The LNSE equations can capture the coupling of acoustics with the flame response in addition to the coupling with vortical and entropy waves

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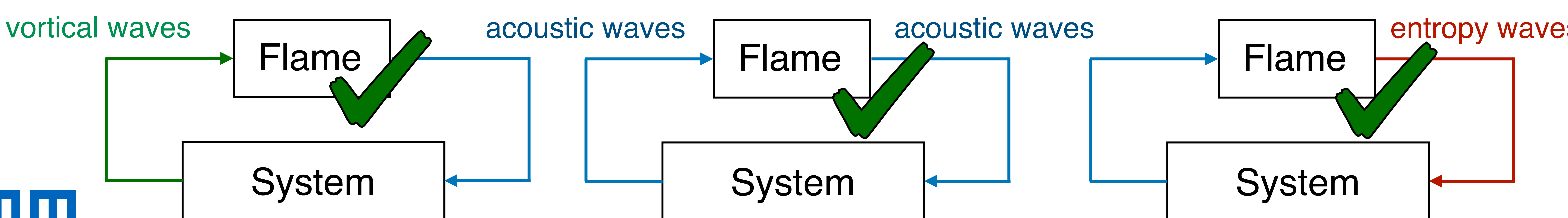

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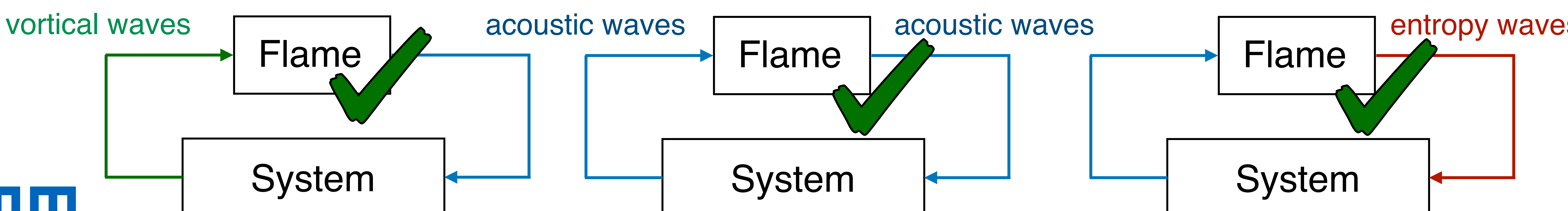

Obtaining and interpreting the solution of the LNSE equations is not straight forward.

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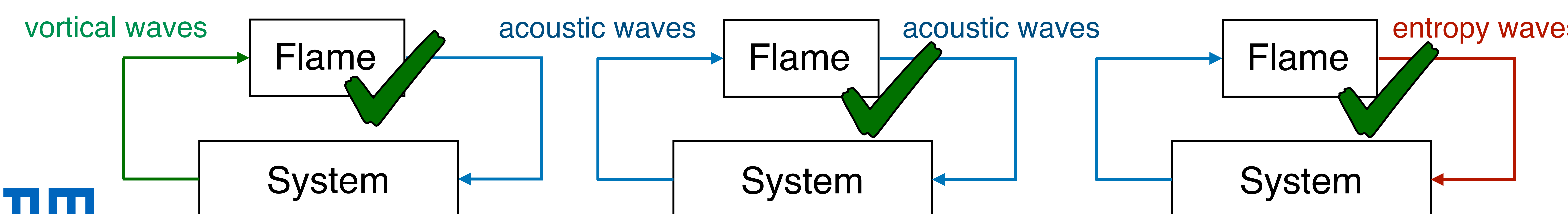

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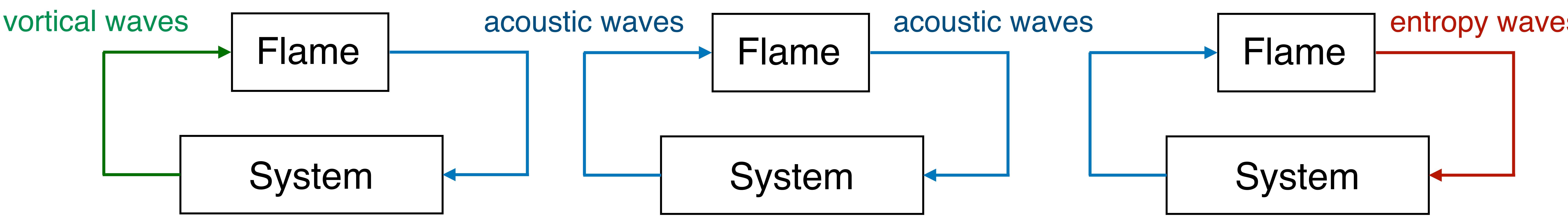

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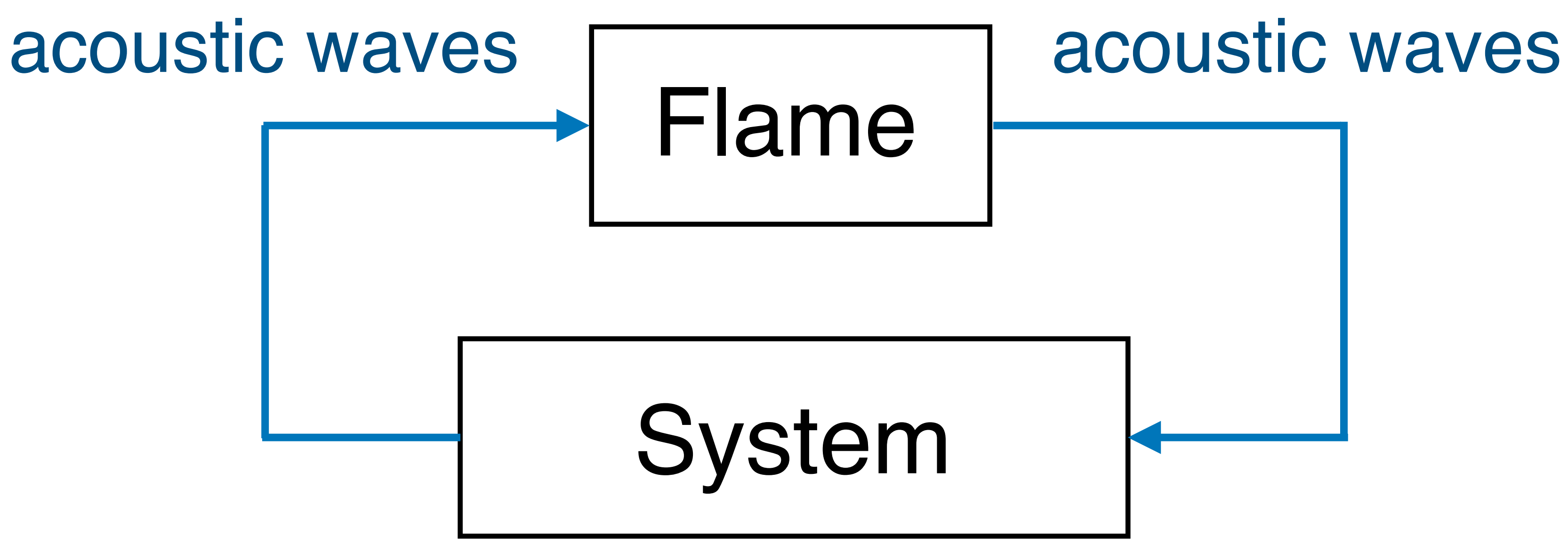
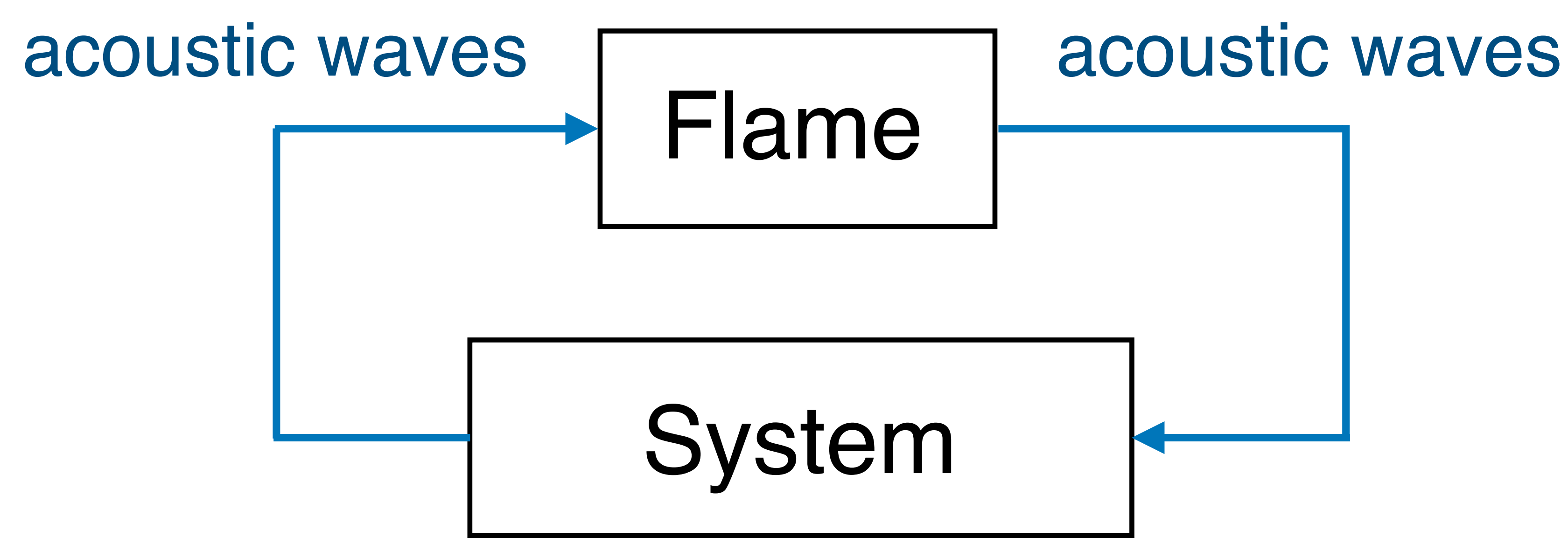


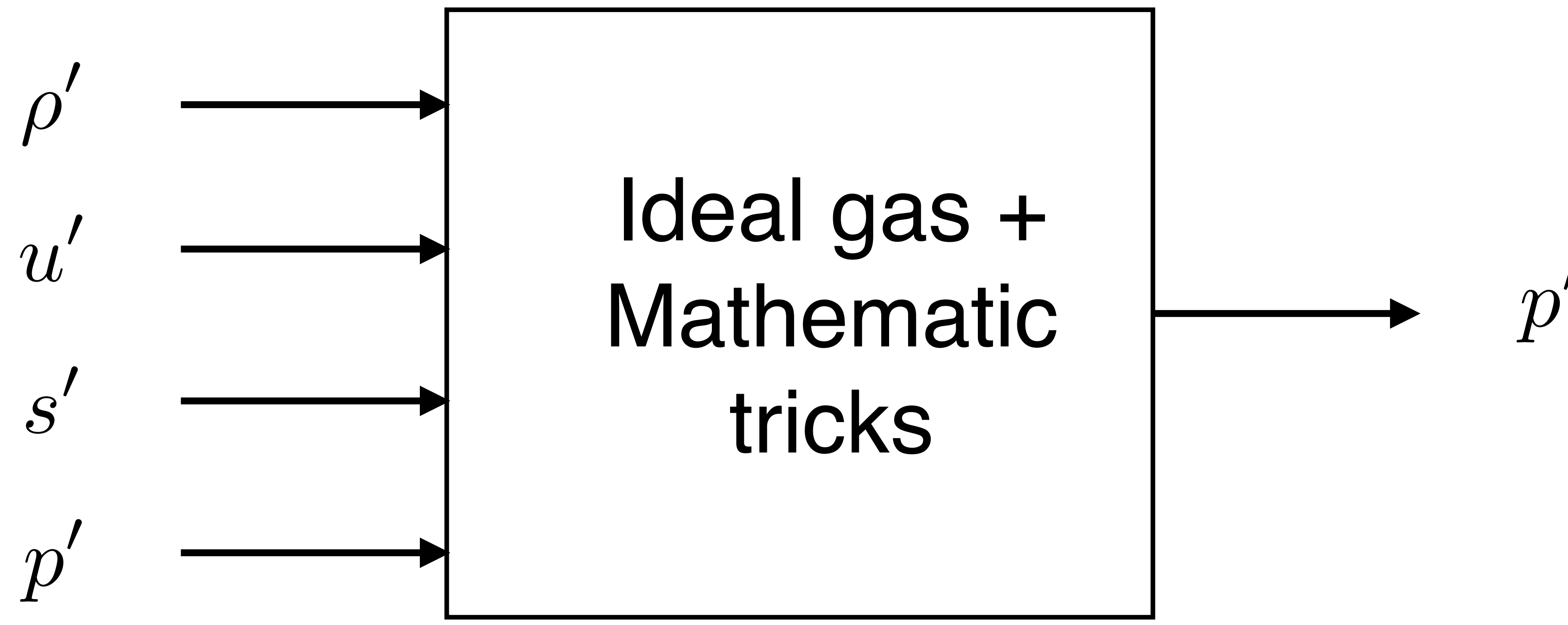
In order to capture the essence of this mechanism, we should consider the information given by the conservation of mass, momentum

The coupling between the flame response and the acoustics of the system is the most investigated mechanism in thermoacoustics

In order to capture the essence of this mechanism, we should consider the information given by the conservation of mass, momentum and energy. However, the acoustic pressure is the only one variable of interest.



Challenge: Compress three equations (four with equation of state) in one.



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Combine Eq. of state and 1st-2nd law of Thermodynamics

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state $$p = \rho R T$$

1st & 2nd law $$c_p T = T\frac{Ds}{Dt} + v\frac{Dp}{Dt}$$

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$$\Rightarrow \quad \frac{1}{c_p}\frac{Ds}{Dt} = \frac{1}{\gamma p}\frac{Dp}{Dt} - \frac{1}{\rho}\frac{D\rho}{Dt}$$

Combine Eq. of mass and energy with Eq. 4

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So far we have reduced the entire system to two equations

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- Mass conservation
- Energy conservation
- ideal gas

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Invent a new variable call π : a pure mathematical trick

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Finally we apply the classical trick for the wave equation to be obtained

$$\frac{\partial}{\partial t}\left[\overbrace{\frac{\partial \pi}{\partial t} + \frac{\partial u_i}{\partial x_i} = \frac{(\gamma - 1)}{\gamma p}\dot{q}}^{\text{momentum}}\right] \quad - \quad \frac{\partial}{\partial x_i}\left[\overbrace{\frac{\partial u_i}{\partial t} = -c^2 \frac{\partial \pi}{\partial x_i} + \frac{1}{\rho}\frac{\tau_{ij}}{\partial x_j}}^{\substack{\text{- Mass conservation} \\ \text{- Energy conservation} \\ \text{- ideal gas}}}\right]$$

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- Energy conservation
- ideal gas

$$\frac{\partial}{\partial t}\overbrace{\left[\frac{\partial \pi}{\partial t} + \frac{\partial u_i}{\partial x_i} = \frac{(\gamma - 1)}{\gamma p}\dot{q}\right]}^{\text{momentum}} - \frac{\partial}{\partial x_i}\overbrace{\left[\frac{\partial u_i}{\partial t} = -c^2\frac{\partial \pi}{\partial x_i} + \frac{1}{\rho}\frac{\tau_{ij}}{\partial x_j}\right]}^{\text{Mass conservation, Energy conservation, ideal gas}}$$

$$\Downarrow$$

$$\boxed{\frac{\partial^2 \pi}{\partial t} - \frac{\partial}{\partial x_i}\left(c^2\frac{\partial \pi}{\partial x_i} + \frac{1}{\rho}\frac{\tau_{ij}}{\partial x_j}\right) = \frac{\partial}{\partial t}\left(\frac{(\gamma - 1)}{\gamma p}\dot{q}\right)}$$

Finally we apply the classical trick for the wave equation to be obtained

- Mass conservation
- Energy conservation
- ideal gas

$$\overbrace{\frac{\partial}{\partial t}\left[\frac{\partial \pi}{\partial t}+\frac{\partial u_i}{\partial x_i}=\frac{(\gamma-1)}{\gamma p}\dot{q}\right]}^{\text{momentum}} \quad - \quad \overbrace{\frac{\partial}{\partial x_i}\left[\frac{\partial u_i}{\partial t}=-c^2\frac{\partial \pi}{\partial x_i}+\frac{1}{\rho}\frac{\tau_{ij}}{\partial x_j}\right]}$$

$$\Downarrow$$

$$\boxed{\frac{\partial^2 \pi}{\partial t}-\frac{\partial}{\partial x_i}\left(c^2\frac{\partial \pi}{\partial x_i}+\cancelto{0}{\frac{1}{\rho}\frac{\tau_{ij}}{\partial x_j}}\right)=\frac{\partial}{\partial t}\left(\frac{(\gamma-1)}{\gamma p}\dot{q}\right)}$$

Very few assumptions have been made so far

So far we have:

- $\dagger$ Assumed ideal gas
- $\dagger$ Assumed low-Mach number flow
- $\dagger$ Assumed viscous terms to be negligible

$$\pi' = \frac{1}{\gamma}(\ln(p))' = \frac{p'}{\gamma \bar{p}}$$

We have not yet linearized. We will do that now $\Rightarrow$

$$p' \ll \bar{p}$$

A wave equation for reacting flows in time and frequency domain is obtained

Wave Equation (time domain)

$$\frac{\partial^2 p'}{\partial t} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial p'}{\partial x_i}\right) = (\gamma - 1)\frac{\partial \dot{q}'}{\partial t}$$

A wave equation for reacting flows in time and frequency domain is obtained

Wave Equation (time domain)

$$\frac{\partial^2 p'}{\partial t} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial p'}{\partial x_i}\right) = (\gamma - 1)\frac{\partial \dot{q}'}{\partial t}$$

Helmholtz Equation (frequency domain)

$$s^2 \hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}$$

By assuming a harmonic decomposition $[]' = \hat{[]} e^{st}$

Laplace variable

$$s = \underbrace{\sigma}_{\text{growth rate}} + i\underbrace{\omega}_{\text{frequency}}$$

The Helmhotz Equation + Flame response = nonlinear Eigenvalue problem

$$\boxed{s^2 \hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}}$$

$$\hat{\dot{q}} = f(\hat{p})$$

and/or

$$\hat{\dot{q}} = f\left(\frac{\partial \hat{p}}{\partial x}\right)$$

A flame response is needed to close the problem

The Helmhotz Equation + Flame response = nonlinear Eigenvalue problem

$$\boxed{s^2\hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2\frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}}$$

Premixed flames are known to respond predominantly to velocity fluctuations (upstream of the flame). We also know that

$\hat{u}$ is function of $\frac{\partial \hat{p}}{\partial x}$

$\Leftarrow$

$$\hat{\dot{q}} = f(\hat{p})$$

and/or

$$\hat{\dot{q}} = f\left(\frac{\partial \hat{p}}{\partial x}\right)$$

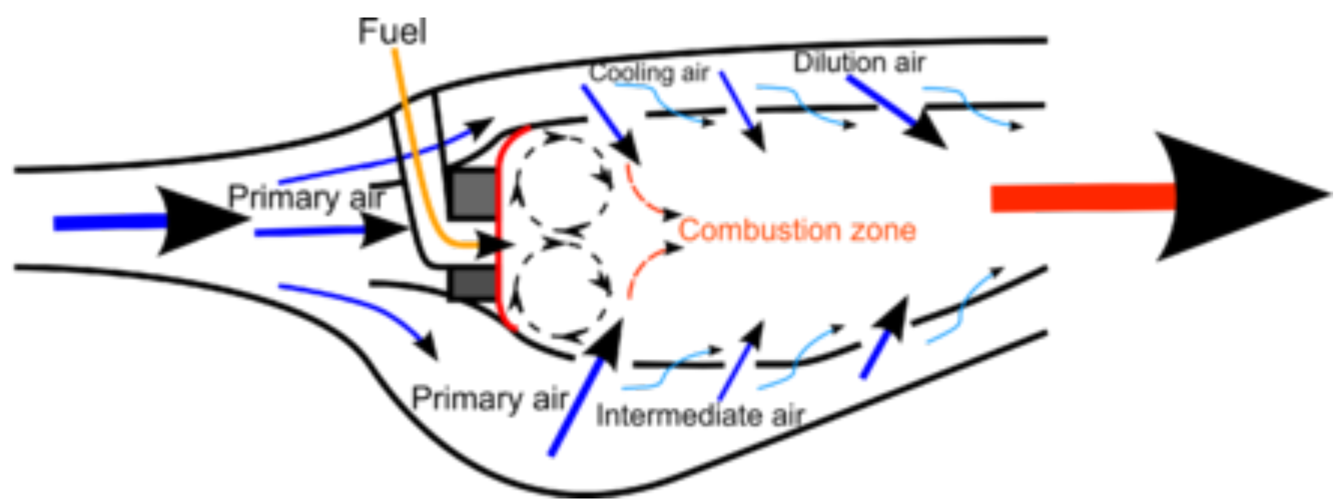
A flame response is needed to close the problem

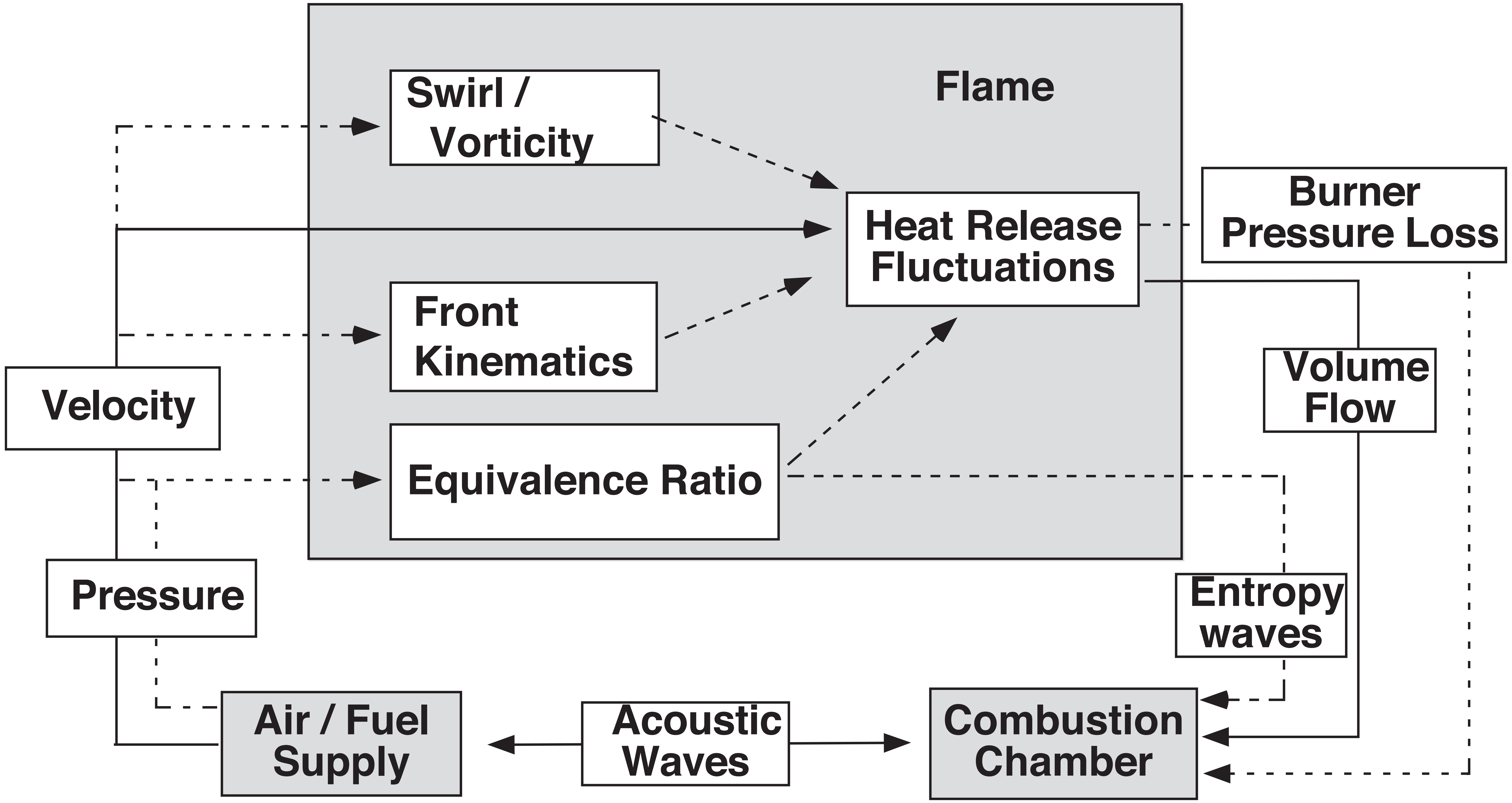
Outline

- From the Navier-Stokes equations to the LRF and LNSE
- From the Navier-Stokes equations to the wave equation
- Recapitulating: What is the Helmholtz Equation good for?

recapitulating

The linearized Navier Stokes equations should be capable of modeling this





The LNSE equations can capture the coupling of acoustics with the flame response in addition to the coupling with vortical and entropy waves

These equations are known as the **Linearized Navier Stokes (LNSE)** equations

mass

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x_j}\left(\bar{\rho} u_j' + \rho' \bar{u}_j\right) = 0$$

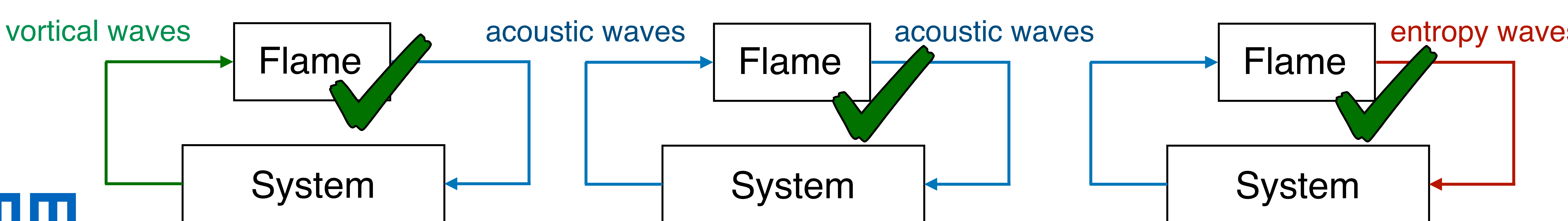
momentum

$$\frac{\partial}{\partial t}\left(\bar{\rho} u_i' + \rho' \bar{u}_i\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho} \bar{u}_i u_j' + \bar{\rho} u_i' \bar{u}_j + \rho' \bar{u}_i \bar{u}_j\right) = -\frac{\partial p'}{\partial x_i} + \frac{\partial \tau_{ij}'}{\partial x_j}$$

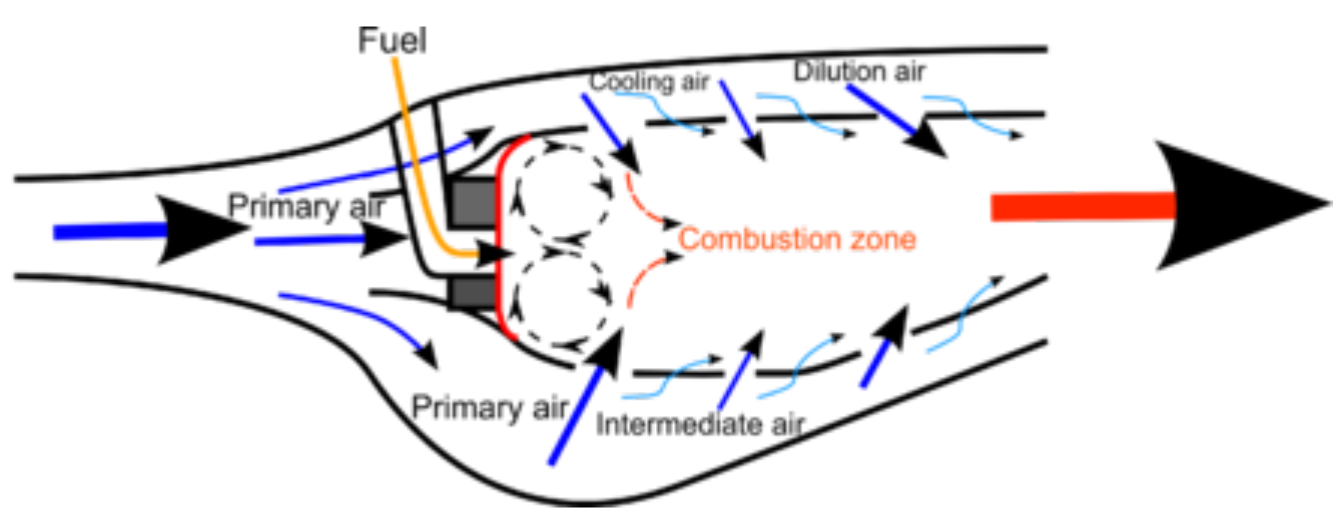
energy

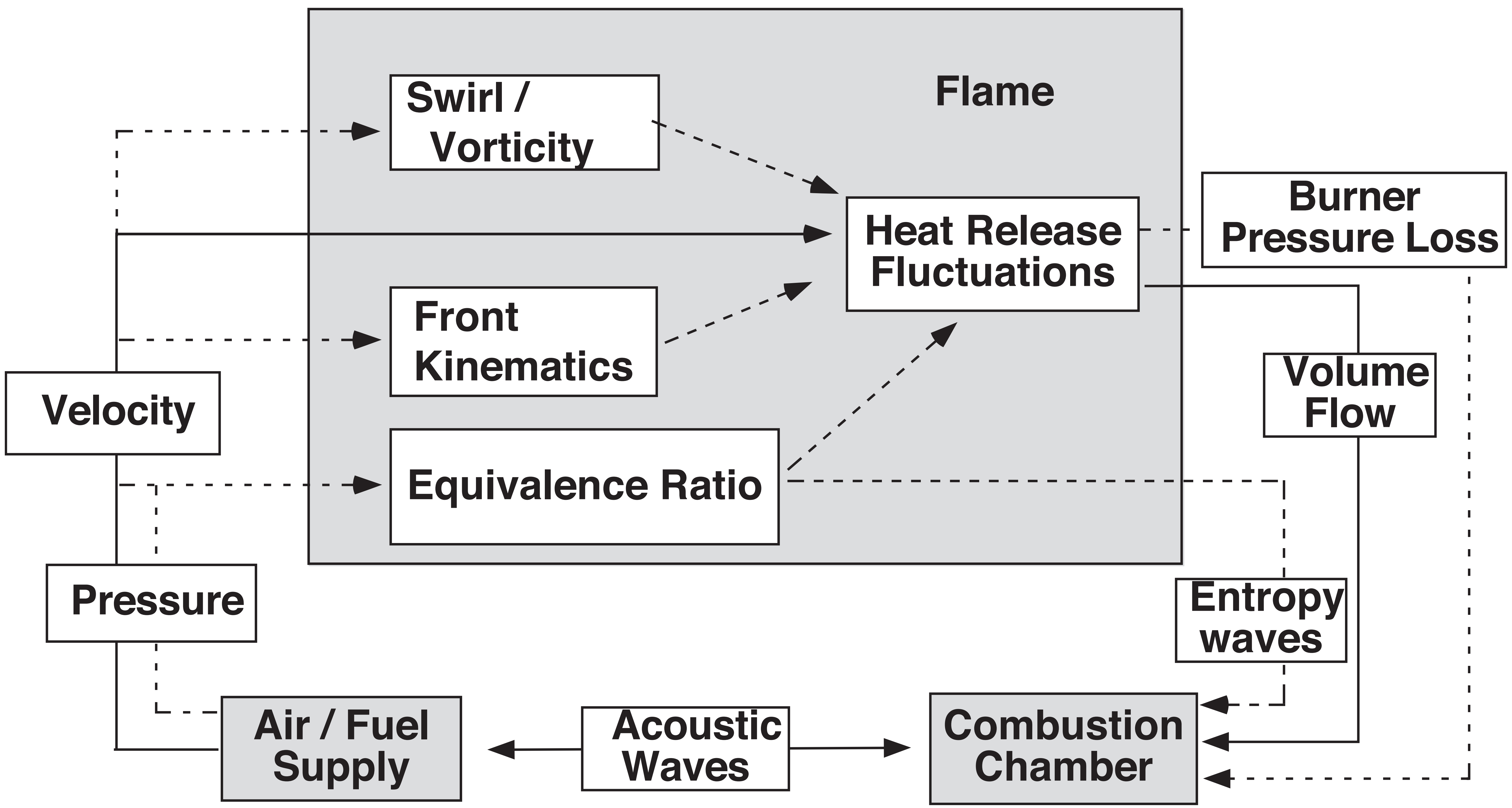
$$\bar{T}\left[\frac{\partial}{\partial t}\left(\bar{\rho} s' + \rho' \bar{s}\right) + \frac{\partial}{\partial x_j}\left(\bar{\rho} \bar{u}_j s' + \bar{\rho} u_j' \bar{s} + \rho' \bar{u}_j \bar{s}\right)\right] + T' \frac{\partial}{\partial x_j}\left(\bar{\rho} \bar{u}_j \bar{s}\right) = \dot{q}'$$

Requires a flame response



The Helmholtz Equation should be capable of modeling only part of this





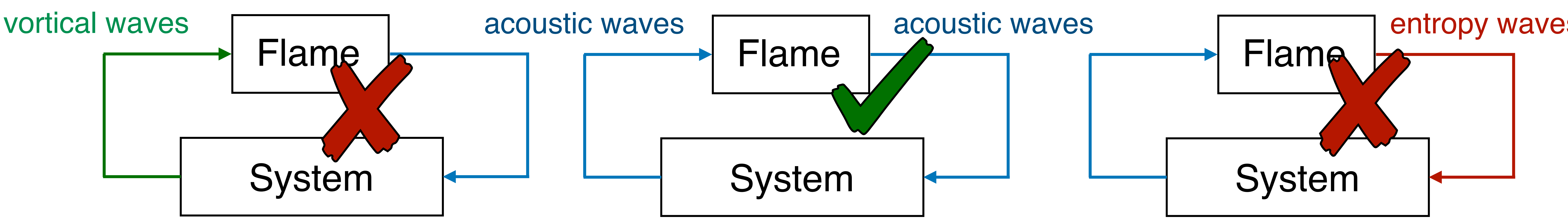
Sattelmayer (1997)

The Helmholtz Equation can only model the interaction between the flame and acoustics

This equation is known as the **Helmholtz** equation

$$s^2\hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}$$

Requires a
flame response

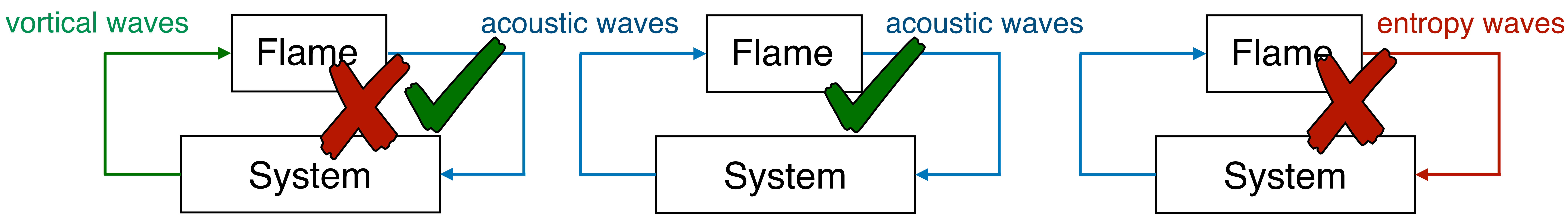


The influence of vortical waves is usually included in the flame response

This equation is known as the **Helmholtz** equation

$$s^2\hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2\frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}$$

Requires a flame response



The energy conversion mechanism (entropy-acoustic) can be modeled by a transfer function as done by Motheau et al. 2014

This equation is known as the **Helmholtz** equation

$$s^2\hat{p} - \frac{\partial}{\partial x_i}\left(\bar{c}^2 \frac{\partial \hat{p}}{\partial x_i}\right) = s(\gamma - 1)\hat{\dot{q}}$$

Requires a flame response