Introduction to Acoustic Analogies

Christophe Schram

von Karman Institute for Fluid Dynamics

Aeronautics and Aerospace
& Environmental and Applied Fluid Dynamics depts

Plan

- Linearized conservation equations
  - Linearized sources of sound
- Lighthill's analogy
  - Non-linear sources of sound
  - Choice of the acoustical variable
- Solving the inhomogeneous wave propagation equation
  - Green's functions (free-field and tailored)
  - Integral solution
- Duct acoustics
  - Tailored Green's functions for rectangular and circular ducts
- Curle's analogy
  - Compact and non-compact sources
- Acoustic energy
  - Cycle-averaged integral form
  - Applications: hard wall and Rijke's tube

Navier-Stokes equations

- Continuity:

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla\cdot\mathbf{v}$$

– (dilatation rate)

rate of change of density moving with the fluid particle

- Newton's law:

Acceleration of the fluid particle

External body forces applied to the fluid particle

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla\mathbf{\Pi} + \mathbf{f}$$

External stresses applied to the fluid particle:

$$\Pi_{ij} = p\,\boldsymbol{\delta}_{ij} - \sigma_{ij}$$

Hydrostatic pressure

Viscous stresses

$$\Rightarrow \quad \rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \nabla\cdot\boldsymbol{\sigma} + \mathbf{f}$$

Linearization

- Continuity and momentum equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = Q_m \qquad \rho \frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{f}$$

- Perturbations = deviations with respect to uniform and stagnant fluid:

$$\rho = \rho_0 + \rho'$$
$$p = p_0 + p'$$
$$\mathbf{v} = \mathbf{v}'$$

- At first order:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m \qquad \rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f}$$

Wave propagation equation

- Eliminate $\mathbf{v}'$ from the linearized conservation equations:

$$\frac{\partial}{\partial t}\left\{\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m\right\}$$

$$-\nabla \cdot \left\{\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f}\right\}$$

$$\Rightarrow \quad \frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = \frac{\partial Q_m}{\partial t} - \nabla \cdot \mathbf{f} - \nabla \cdot (\nabla \cdot \boldsymbol{\sigma}')$$

- Introduce constitutive equation: $p = p(\rho, s)$

$$\Rightarrow \quad p' = \left(\frac{\partial p}{\partial \rho}\right)_s \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s' \simeq c_0^2\, \rho'$$

Sources of sound (linearized)

D'Alembertian

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \frac{\partial Q_m}{\partial t} + \frac{1}{c_0^2}\left(\frac{\partial p}{\partial s}\right)_\rho \frac{\partial^2 s'}{\partial t^2} - \nabla \cdot \mathbf{f} - \nabla \cdot (\nabla \cdot \boldsymbol{\sigma}')$$

Fluctuating mass injection

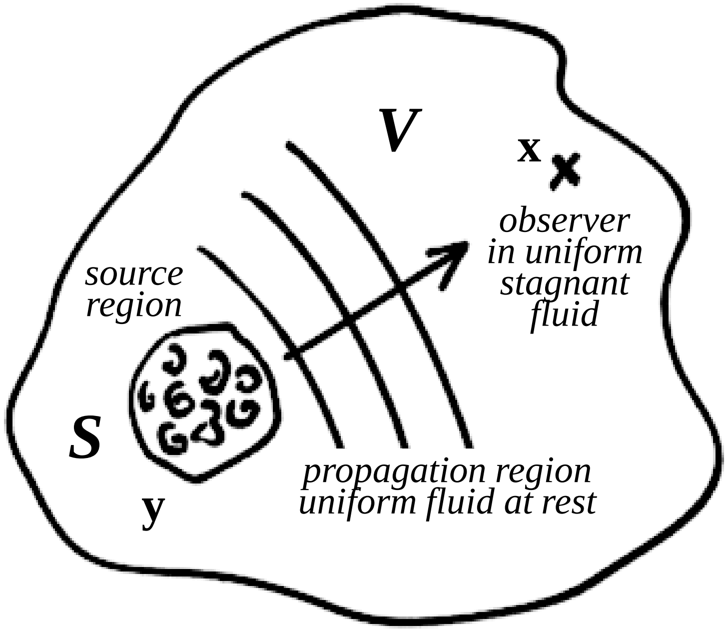
Entropy fluctuations

Non-uniform force field

Fluctuating viscous stresses

In a non-linearized context: Lighthill’s aeroacoustical analogy

- The problem of sound produced by a turbulent flow is, **from the listener’s point of view**, analogous to a problem of propagation in a uniform medium at rest in which equivalent sources are placed.



- Wave propagation region: linear wave operator applies

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = 0$$

No source

- Turbulent region: non-linear fluid mechanics equations apply

<u>continuity</u>

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = 0$$

<u>momentum</u>

$$\frac{\partial \rho v_i}{\partial t} + \frac{\partial \rho v_i v_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j}$$

Lighthill's analogy: formal derivation

$$\frac{\partial}{\partial t}\left\{\frac{\partial \rho}{\partial t}+\frac{\partial \rho v_i}{\partial x_i}=0\right\}$$

$$-\frac{\partial}{\partial x_i}\left\{\frac{\partial \rho v_i}{\partial t}+\frac{\partial \rho v_i v_j}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\frac{\partial \sigma_{ij}}{\partial x_j}\right\}$$

$$\Rightarrow \quad \frac{\partial^2 \rho}{\partial t^2}=\frac{\partial^2\left(\rho v_i v_j-\sigma_{ij}\right)}{\partial x_i x_j}+\frac{\partial^2 p}{\partial x_i^2}$$

$$\frac{\partial^2 \rho}{\partial t^2}\boxed{-c_0^2\,\frac{\partial^2 \rho}{\partial x_i^2}}=\frac{\partial^2\left(\rho v_i v_j-\sigma_{ij}\right)}{\partial x_i x_j}+\frac{\partial^2\left(p\boxed{-c_0^2\,\rho}\right)}{\partial x_i^2}$$

Reference state

- Reformulation of fluid mechanics equations:

$$\frac{\partial^2 \rho}{\partial t^2} - c_0^2 \frac{\partial^2 \rho}{\partial x_i^2} = \frac{\partial^2 \left(\rho v_i v_j - \sigma_{ij}\right)}{\partial x_i x_j} + \frac{\partial^2 \left(p - c_0^2\, \rho\right)}{\partial x_i^2}$$

- Definition of a reference state:

$$\rho \equiv \rho_0 + \rho'$$

$$p \equiv p_0 + p'$$

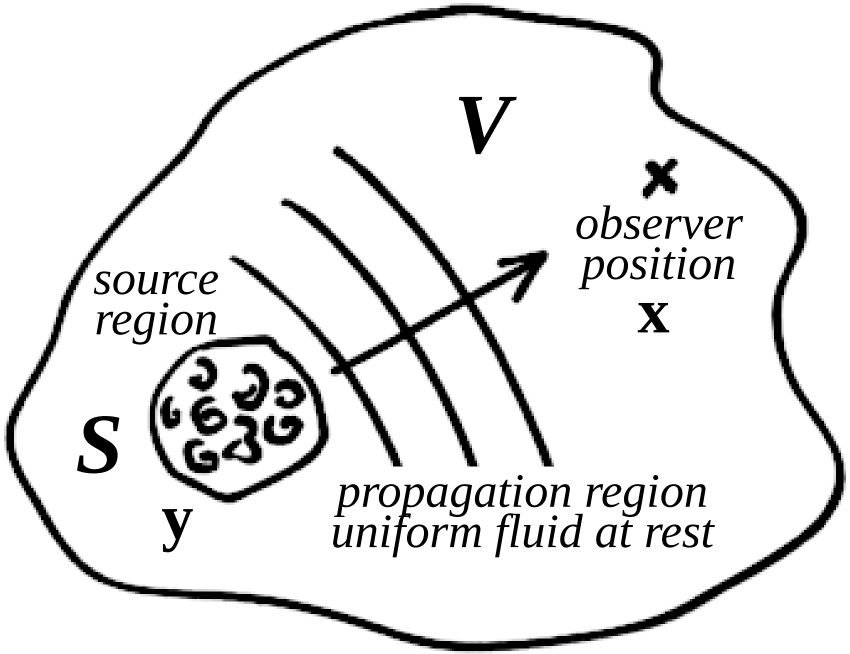
$$v_i \equiv v_i'$$

- Aeroacoustical analogy :

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2} = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

$$T_{ij} = \rho v_i v_j + \left(p' - c_0^2\, \rho'\right) \boldsymbol{\delta}_{ij} - \sigma_{ij}$$

Lighthill's tensor



Choice of the acoustic variable

- Combining the mass and momentum equations yields:

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x_i x_j} \left(\rho v_i v_j - \sigma_{ij} \right) + \frac{\partial^2 p}{\partial x_i^2}$$

- From there, two choices are possible for the acoustic variable:
  - Acoustical density perturbation:

$$\frac{\partial^2 \rho'}{\partial t^2} \boxed{- c_0^2 \frac{\partial^2 \rho'}{\partial x_i^2}} = \frac{\partial^2}{\partial x_i x_j} \left(\rho v_i v_j - \sigma_{ij} \right) + \frac{\partial^2}{\partial x_i^2} \left(p' \boxed{- c_0^2 \rho'} \right)$$

Isentropic noise generation

  - Acoustical pressure perturbation:

$$\boxed{\frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2}} - \frac{\partial^2 p'}{\partial x_i^2} = \frac{\partial^2}{\partial x_i x_j} \left(\rho v_i v_j - \sigma_{ij} \right) + \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \left(\boxed{p'} - c_0^2 \rho' \right)$$

Combustion noise

Solving the wave equation

- General form of the wave equation:

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \, \nabla^2 \rho' = q(\mathbf{x}, t) \qquad\qquad \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = q(\mathbf{x}, t)$$

- Homogeneous solution:

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \, \nabla^2 \rho' = 0 \qquad\qquad \frac{1}{c_0^2} \frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$$

$\Rightarrow$ acoustic field driven by initial and boundary conditions

- In frequency domain, at pulsation ω : Helmholtz equation

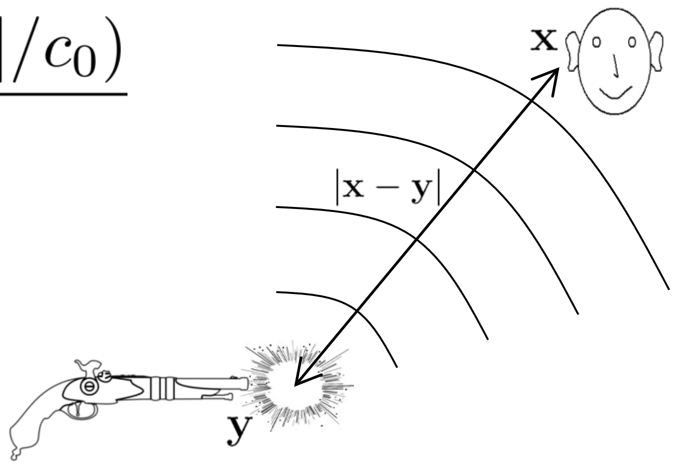
$$\nabla^2 p + k^2 \, p = 0 \qquad\qquad p'(\mathbf{x}, t) = p(\mathbf{x}) \, \mathrm{e}^{\mathrm{i}\omega t}$$

Green's function

- Wave equation: $$\frac{\partial^2 G}{\partial t^2} - c_0^2 \nabla^2 G = \delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)$$ + BCs

- Free-field boundary conditions (Sommerfeld):

$$G_0(\mathbf{x},t|\mathbf{y},\tau) = \frac{\delta(t-\tau-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x}-\mathbf{y}|}$$



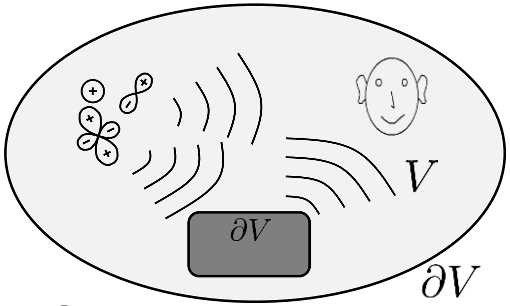
- Retarded (emission) time:

$$\tau^* = t - |\mathbf{x}-\mathbf{y}|/c_0$$

- Useful properties:
  - Dirac function → convenient to obtain an integral solution
  - Reciprocity: $G(\mathbf{x},t|\mathbf{y},\tau) = G(\mathbf{y},-\tau|\mathbf{x},t)$

Integral solution of the wave eq.

- Our problem to solve: $$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \, \nabla^2 \rho' = q(\mathbf{x}, t)$$
- Green's function definition: $$\frac{\partial^2 G}{\partial t^2} - c_0^2 \, \nabla^2 G = \delta(\mathbf{x} - \mathbf{y}) \, \delta(t - \tau)$$
- 'After some algebra':



$$\rho'(\mathbf{x}, t) = \int_{t_0}^{t} \iiint_V q(\mathbf{y}, \tau) \, G\left(\mathbf{x}, t | \mathbf{y}, \tau\right) \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau$$
$$- c_0^2 \int_{t_0}^{t} \iint_{\partial V} \left(\rho'(\mathbf{y}, \tau) \, \frac{\partial G}{\partial y_i} - G \, \frac{\partial \rho'(\mathbf{y}, \tau)}{\partial y_i} \right) n_i \, \mathrm{d}^2\mathbf{y} \mathrm{d}\tau$$

What if $\frac{\partial G}{\partial n} = 0$?

$$c_0^2 \, \frac{\partial \rho'}{\partial n} = -\mathrm{i}\omega \rho_0 v_n$$

- Further simplifications:
  - No solid surface in the propagation region, or
  - Non-vibrating surfaces and tailored Green's function

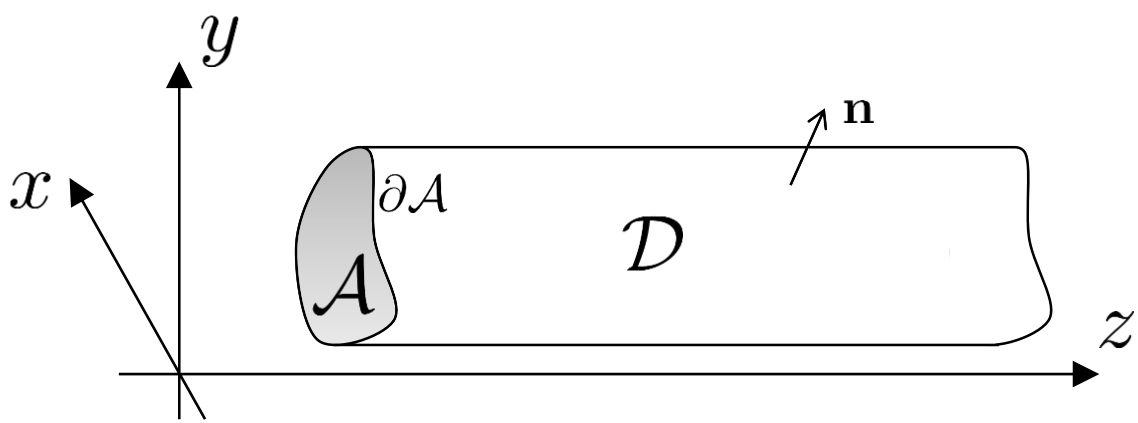
Integrating is good for health!

$$\rho'(\mathbf{x}, t) = \int_{t_0}^{t} \iiint_V q(\mathbf{y}, \tau) \, G\left(\mathbf{x}, t | \mathbf{y}, \tau\right) \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau$$

Tailored Green functions

- In a few cases: analytical Green's functions
  - Infinite planes: image sources (semi-anechoic environment)
  - Semi-infinite plane (trailing edge noise)
  - Infinite straight ducts: rectangular, cylindrical, annular
- In other cases: semi-analytical Green's functions
  - Compact (low-frequency) Green's functions (Howe)
  - Wiener-Hopf technique, Schwarzchild's technique (TE-LE backscattering, Roger)
  - Slowly-varying duct (Rienstra)
- In all other cases: numerical Green's functions
  - Low-frequency techniques
    - Finite Element Methods, Boundary Element Methods
  - High-frequency techniques
    - Ray-tracing methods, Statistical Energy Analysis
  - Mid-frequency techniques
    - Multigrid techniques, fast multipole BEM, …

Duct acoustics in frequency domain



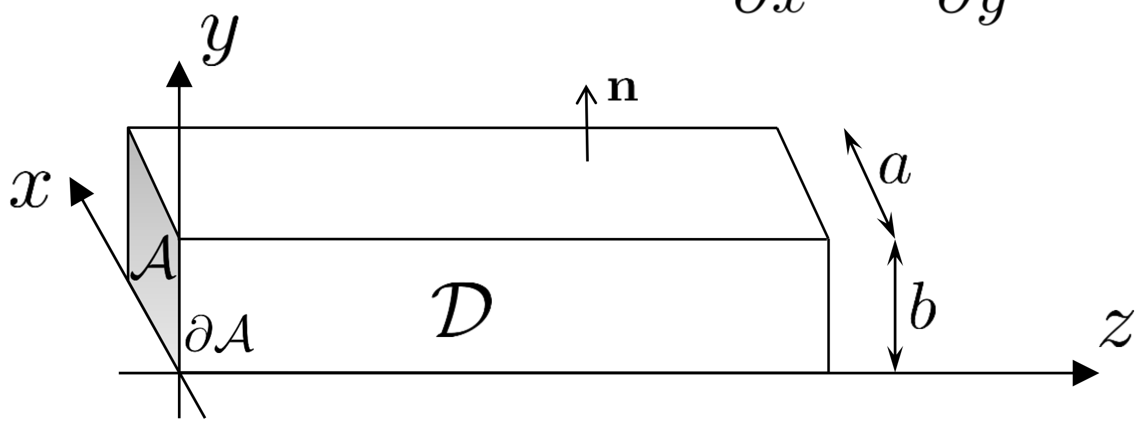
- Fourier decomposition:
$$p'(x,y,z,t) = p(x,y,z)\,\mathrm{e}^{\mathrm{i}\omega t}$$
$$\mathbf{v}'(x,y,z,t) = \mathbf{v}(x,y,z)\,\mathrm{e}^{\mathrm{i}\omega t}$$

- Homogeneous Helmholtz equation: $\nabla^2 p + k^2 p = 0$
with the wavenumber $k = \omega/c_0$

- Neumann BC on $\partial\mathcal{A}$: $\mathrm{i}\omega\mathbf{v} + \nabla p = 0$

Rectangular duct

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$



- Separation of variables: $p(x, y, z) \equiv F(x)\, G(y)\, H(z)$
- PDE → set of ODEs:

$$\partial^2 F/\partial x^2 = -\alpha^2 F$$
$$\partial^2 G/\partial y^2 = -\beta^2 G$$
$$\partial^2 H/\partial z^2 = -(k^2 - \alpha^2 - \beta^2)\, H$$

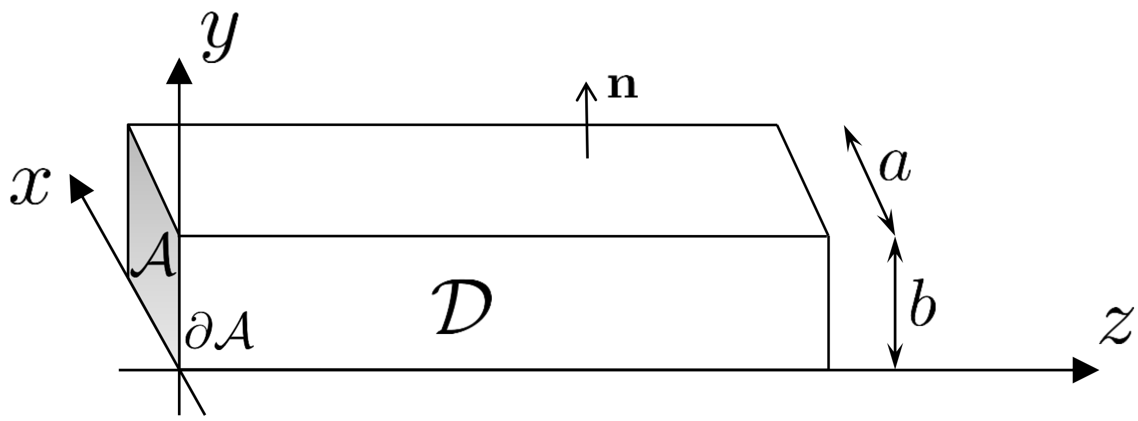
+ Neumann BCs

$$F(x) = \cos(\alpha_n x)\,,\ \ \alpha_n = n\pi/a\,,\ \ n = 0,\, 1,\, 2,\, \ldots$$
$$G(y) = \cos(\beta_m y)\,,\ \ \beta_m = m\pi/b\,,\ \ m = 0,\, 1,\, 2,\, \ldots$$
$$H(z) = \mathrm{e}^{\mp \mathrm{i} k_{nm} z}\,,\ \ k_{nm} = \sqrt{k^2 - \alpha_n^2 - \beta_m^2}$$

$$\mathrm{Re}(k_{nm}) \geq 0$$
$$\mathrm{Im}(k_{nm}) \leq 0$$

$$p(x, y, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \cos(\alpha_n x)\ \cos(\beta_m y)\ \left(A_{nm}\, \mathrm{e}^{-\mathrm{i} k_{nm} z} + B_{nm}\, \mathrm{e}^{+\mathrm{i} k_{nm} z}\right)$$

Cut-on vs. cut-off modes



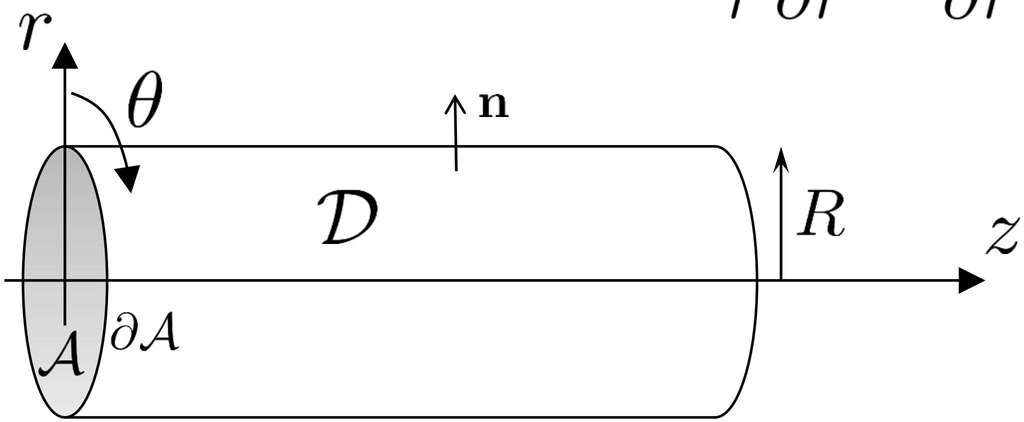
- General solution:

$$p(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \cos(\alpha_n x)\,\cos(\beta_m y)\,\left(A_{nm}\,\mathrm{e}^{-\mathrm{i}k_{nm}z} + B_{nm}\,\mathrm{e}^{+\mathrm{i}k_{nm}z}\right)$$

- Mode wavenumber: $k_{nm} = \sqrt{k^2 - \alpha_n^2 - \beta_m^2}$ $\quad \mathrm{Re}(k_{nm}) \geq 0$, $\mathrm{Im}(k_{nm}) \leq 0$
- For a given mode (n,m):
  - At low frequencies → $k^2 < \alpha_n^2 + \beta_m^2$ → k_{nm} is negative imaginary → cut-off mode evanescent
  - At high frequencies → $k^2 > \alpha_n^2 + \beta_m^2$ → k_{nm} is positive real → cut-on mode propagative
- Planar mode (0,0): always propagative

Circular duct

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$



- Separation of variables: $p(r,\theta,z) \equiv F(z)\,G(r)\,H(\theta)$

$$\partial^2 H/\partial\theta^2 = -m^2 H$$

- PDE → set of ODEs: (+ Neumann BC @ $r = R$)

$$\partial^2 G/\partial r^2 + (1/r)\,\partial G/\partial r = (m^2/r^2 - \alpha^2)\,G$$

$$\partial^2 F/\partial z^2 = (\alpha^2 - k^2)\,F$$

$$H(\theta) = \mathrm{e}^{-\mathrm{i}m\theta}\,,\ m = 0,\ \pm 1,\ \pm 2,\ \ldots$$

$$G(r) = \mathrm{J}_m(\alpha_{m\mu} r)\,,\ \mu = 1,\ 2,\ \ldots$$

$$F(z) = \mathrm{e}^{\mp \mathrm{i} k_{m\mu} z}\,,\ k_{m\mu} = \sqrt{k^2 - \alpha_{m\mu}^2}$$

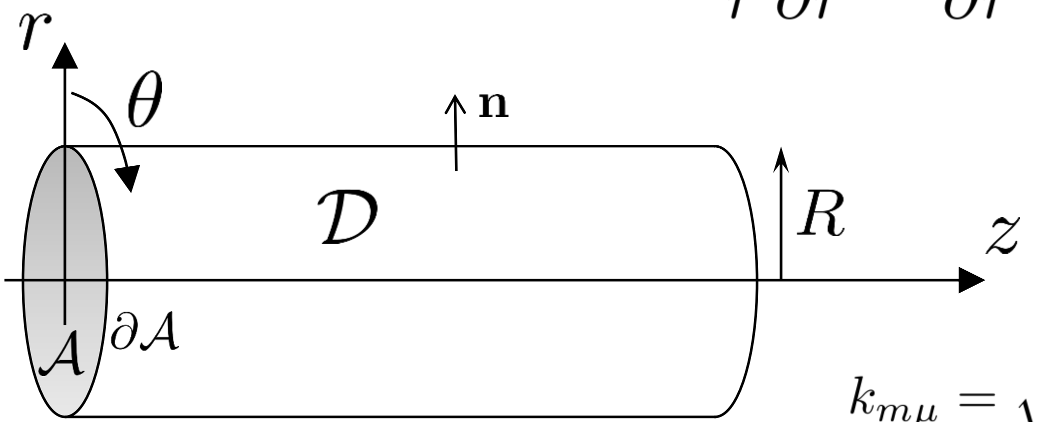
$$\mathrm{Re}(k_{m\mu}) \geq 0$$

$$\mathrm{Im}(k_{m\mu}) \leq 0$$

where $\alpha_{m\mu}$ is the μ-th non-negative, non-trivial solution of $J'_m(\alpha_{m\mu} R) = 0$

Circular duct

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$



$$k_{m\mu} = \sqrt{k^2 - \alpha_{m\mu}^2}$$

$$\mathrm{Re}(k_{m\mu}) \geq 0$$
$$\mathrm{Im}(k_{m\mu}) \leq 0$$

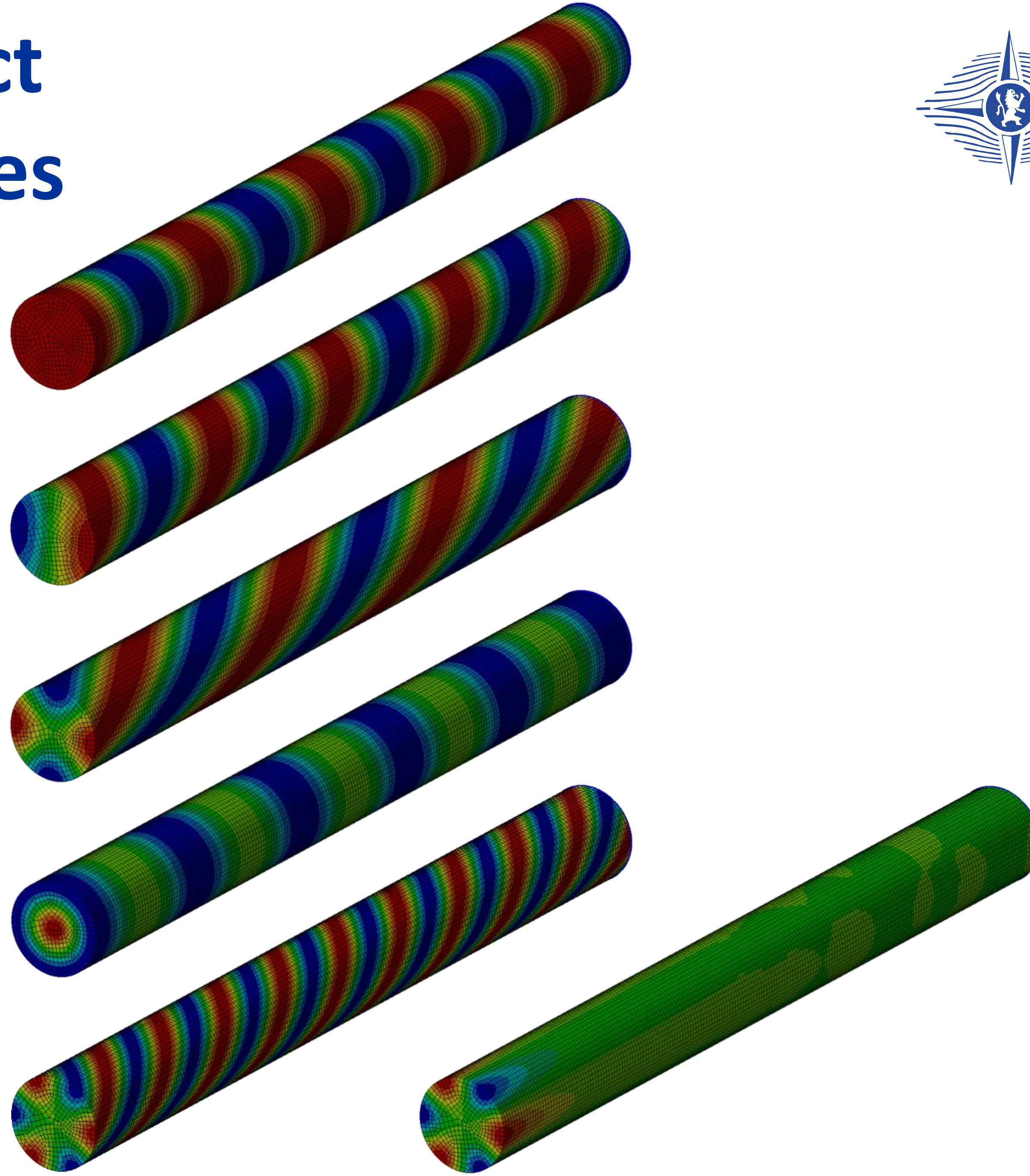
- General solution:

$$p(r,\theta,z) = \sum_{m=-\infty}^{\infty} \sum_{\mu=1}^{\infty} U_{m\mu}(r) \left(A_{m\mu}\,\mathrm{e}^{-\mathrm{i}k_{m\mu}z} + B_{m\mu}\,\mathrm{e}^{+\mathrm{i}k_{m\mu}z}\right) \mathrm{e}^{-\mathrm{i}m\theta}$$

$$U_{m\mu}(r) = N_{m\mu}\,\mathrm{J}_m(\alpha_{m\mu} r) \qquad N_{m\mu} = \left[\frac{1}{2}\left(1 - \frac{m^2}{\alpha_{m\mu}^2}\right)\mathrm{J}_m(\alpha_{m\mu}R)^2\right]^{-1/2}$$

- Evanescent vs. propagative modes (m,µ): same as for rectangular duct
- Planar mode (0,0): always propagative
- First cut-on transversal mode (±1,1): $kR = 1.84$ → $f = \dfrac{99.57\ \mathrm{m/s}}{R} \sim \dfrac{100\ \mathrm{m/s}}{R}$

Circular duct mode shapes



mode (0,0)

mode (1,0)

mode (2,0)

mode (0,1)

mode (3,0)

Curle's analogy: when there's no tailored Green function available

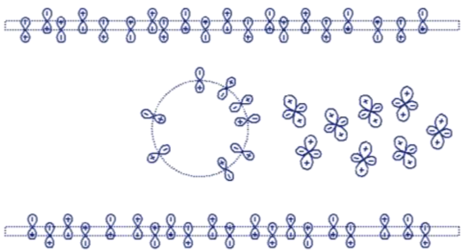
- Lighthill's aeroacoustical analogy: $\dfrac{\partial^2 \rho'}{\partial t^2} - c_0^2 \dfrac{\partial^2 \rho'}{\partial x_i^2} = \dfrac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$
- Integral solution using Green's function

$$\rho'(\mathbf{x},t) = \int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau$$

incident field

$$- c_0^2 \int_{-\infty}^{t} \iint_{\partial V} \left(\rho' \frac{\partial G}{\partial y_i} - G \frac{\partial \rho'}{\partial y_i} \right) n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau$$

scattered field



- Partial integration of source integral

$$\int_{-\infty}^{t} \iiint_V \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} G \, \mathrm{d}^3\mathbf{y} \mathrm{d}\tau = \int_{-\infty}^{t} \iiint_V T_{ij} \frac{\partial^2 G}{\partial y_i \partial y_j} \mathrm{d}^3\mathbf{y} \, \mathrm{d}\tau$$

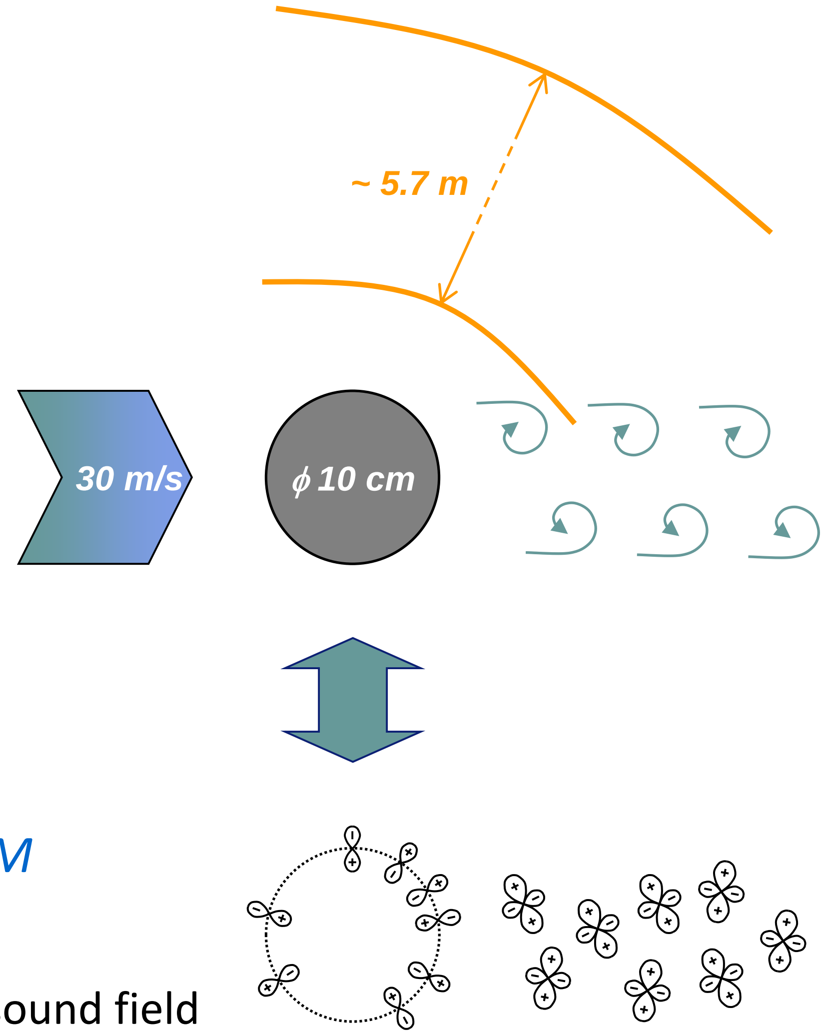
$$+ \int_{-\infty}^{t} \iint_{\partial V} \left\{ \left(-\frac{\partial \rho v_i}{\partial \tau} - \underline{c_0^2 \frac{\partial \rho'}{\partial y_i}} \right) G - \left(\rho v_i v_j + (p' - \underline{c_0^2 \rho'}) \, \delta_{ij} + \sigma_{ij} \right) \frac{\partial G}{\partial y_j} \right\} n_i \, \mathrm{d}^2\mathbf{y} \, \mathrm{d}\tau$$

- Curle's analogy: uses free field Green's function $G_0(t, \mathbf{x}|\tau, \mathbf{y}) = \dfrac{\delta(t - \tau - |\mathbf{x} - \mathbf{y}|/c_0)}{4\pi c_0^2 |\mathbf{x} - \mathbf{y}|}$

$$\Rightarrow \rho'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\frac{T_{ij}}{4\pi c_0^2 |\mathbf{x} - \mathbf{y}|} \right] \mathrm{d}^3\mathbf{y} - \frac{\partial}{\partial x_i} \iint_{\partial V} \left[\frac{p' n_i}{4\pi c_0^2 |\mathbf{x} - \mathbf{y}|} \right] \mathrm{d}^2\mathbf{y}$$

Curle's analogy applied to compact sources

- Sound generated by obstacle in flow
  - Side mirror
  - Antenna
  - Sunroof spoiler
  - HVAC vent grids
  - Landing gear
- Acoustical compactness expressed by Helmholtz number: $He = 2\pi\, f\, D\, /\, c_0$
- Flow unsteadiness expressed by Strouhal number: $Sr = f\, D\, /\, U = \mathrm{O}(0.1\text{-}1)$
- Acoustical compactness depends on Mach number: $He = 2\pi\, Sr\, U\, /\, c_0 = 2\pi\, Sr\, M$
- $He \ll 1$:
  - Compact body, does not scatter its own sound field
  - Neglecting scattering integral is not a significant error



A popular formulation for many industrial applications

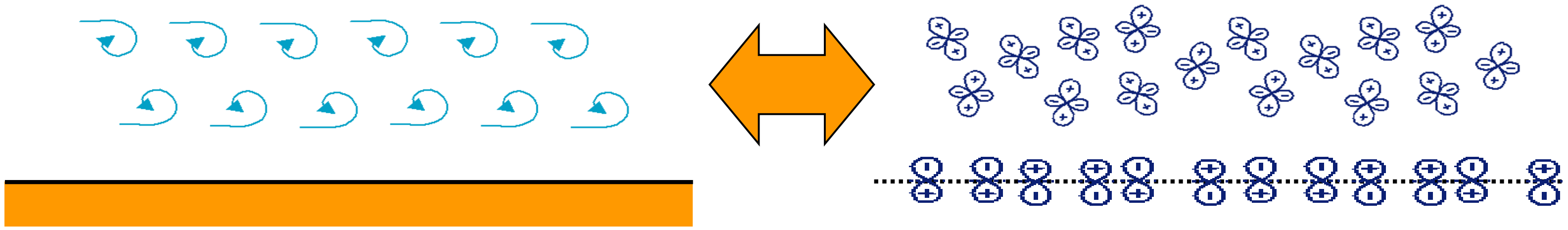
- Curle's formulation is quite powerful
  - It enforces the correct radiation pattern of each source component: $P_{\text{quadru}} / P_{\text{dipo}} \sim M^2$
  - At low Mach numbers, dipolar contribution dominates the quadrupolar one for compact sources
  - Surface scalar (p') data are much less demanding in memory than volumetric, tensorial (T_{ij}) data
  - Surface mesh often available from design stage

$$4\pi c_0^2 \, \rho'(\mathbf{x}, t) = \underbrace{\frac{\partial^2}{\partial x_i \partial x_j} \iiint_V \left[\frac{T_{ij}}{|\mathbf{x} - \mathbf{y}|} \right] \mathrm{d}^3\mathbf{y}}_{\text{Quadrupole, } W \propto M^8 \text{ in free field}} + \underbrace{\frac{\partial}{\partial x_i} \iint_{\partial V} \left[\frac{p' n_i}{|\mathbf{x} - \mathbf{y}|} \right] \mathrm{d}^2\mathbf{y}}_{\text{Dipole, } W \propto M^6 \text{ in free field}}$$

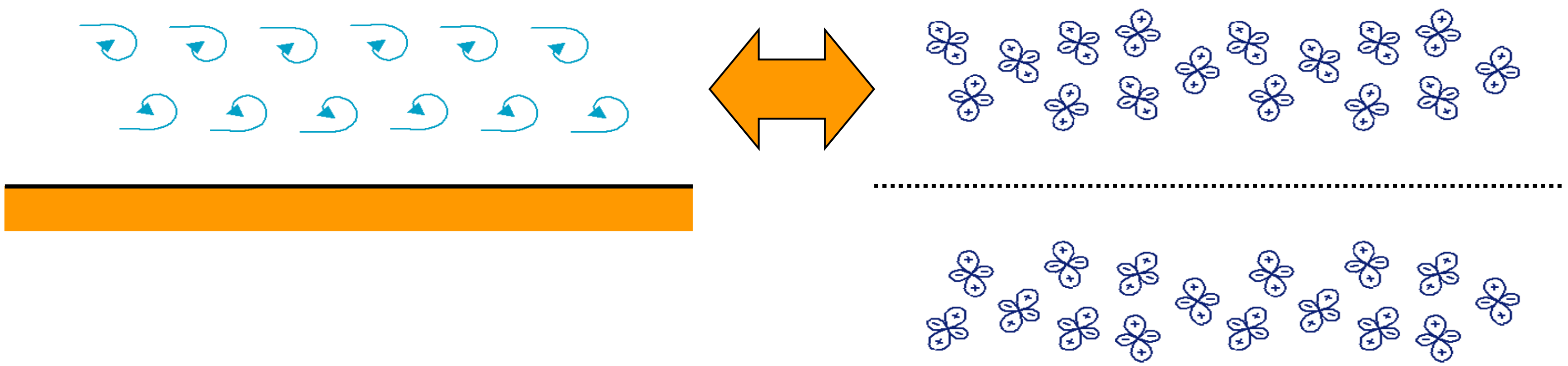
- BUT: tricky implementation for non-compact geometries…

Flat plate boundary layer noise

- Interaction of turbulent boundary layer with an infinite flat plate – can be resolved by two means:
  - Curle's analogy

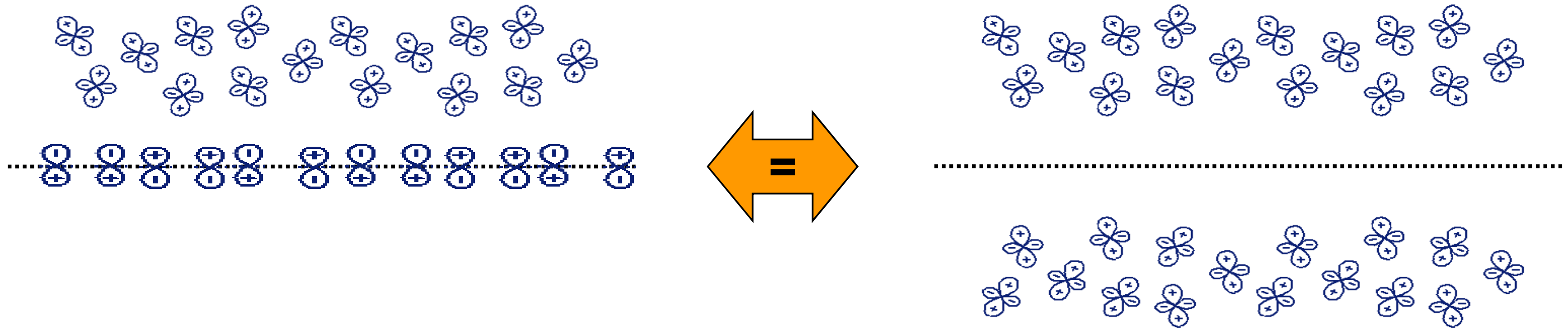


  - Method of images: the effect of the infinite plane can be accounted for by distributing image quadrupoles



Flat surface: dipoles = reflection of the quadrupoles

- Both solutions are equivalent → the dipoles represent the reflection of the quadrupoles (actually, of their wall-normal component only)



- In this case, the dipoles have at most the same acoustic efficiency as the wall-normal component of the quadrupoles

Acoustical energy and intensity

- From the linearized Navier-Stokes equations:

$$\frac{p'}{\rho_0}\left\{\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = Q_m\right\}$$

$$+\mathbf{v}' \cdot \left\{\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' + \nabla \cdot \boldsymbol{\sigma}' + \mathbf{f}\right\}$$

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{2}\rho_0 |\mathbf{v}'|^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c_0^2}\right)}_{E} + \nabla \cdot \underbrace{(p'\mathbf{v}')}_{\mathbf{I}} = \mathbf{v}' \cdot \mathbf{f} + \frac{p' Q_m}{\rho_0}$$

acoustical energy *acoustical intensity*

CAUTION !!!

- We deduced an equation for quadratic functions of perturbation (E and I) from a linear approximation

- These definitions of energy and intensity are only valid in a uniform and stagnant fluid

Steady harmonic oscillations

- Harmonic oscillations at pulsation ω :

$$\int_0^{2\pi/\omega} \iiint_V \left(\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{I} \right) \mathrm{d}t\mathrm{d}V$$
$$= \int_0^{2\pi/\omega} \iiint_V \left(\mathbf{v}' \cdot \mathbf{f} + \frac{p' Q_m}{\rho_0} \right) \mathrm{d}t\mathrm{d}V$$

- Acoustic power radiated over one cycle:

$$\langle P \rangle = \iint_S \langle \mathbf{I} \cdot \mathbf{n} \rangle \, \mathrm{d}S = \iiint_V \left\langle \mathbf{v}' \cdot \mathbf{f} + \frac{p' Q_m}{\rho_0} \right\rangle \mathrm{d}V$$

- Control surface in acoustic far-field: $\langle P \rangle = \iint_S \left\langle \frac{p'^2}{\rho_0 c_0} \right\rangle \mathrm{d}S$

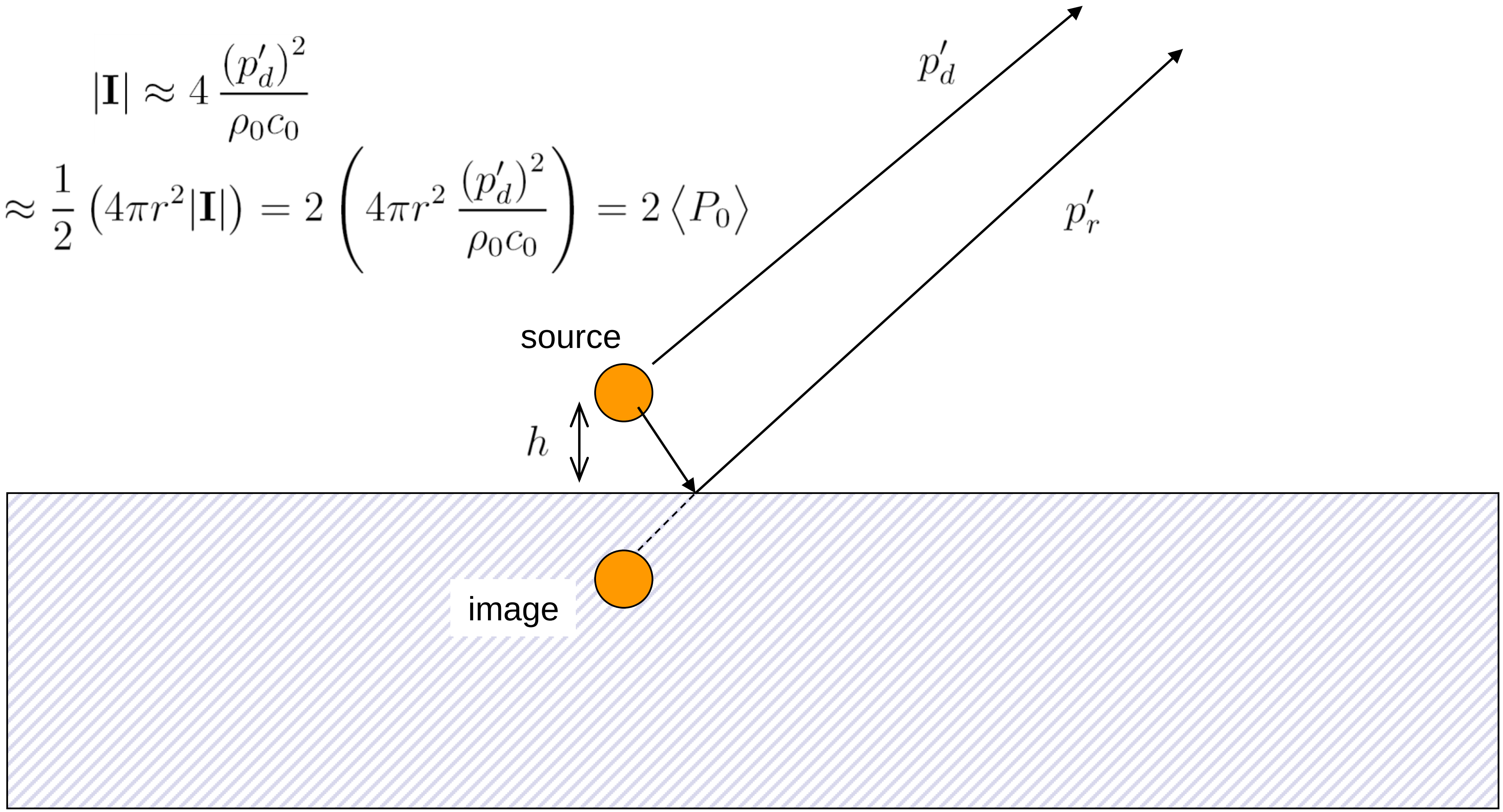
1st example: effect of hard wall

$$kh \ll 1 \;\Rightarrow\; p' = p'_d + p'_r \approx 2\,p'_d$$

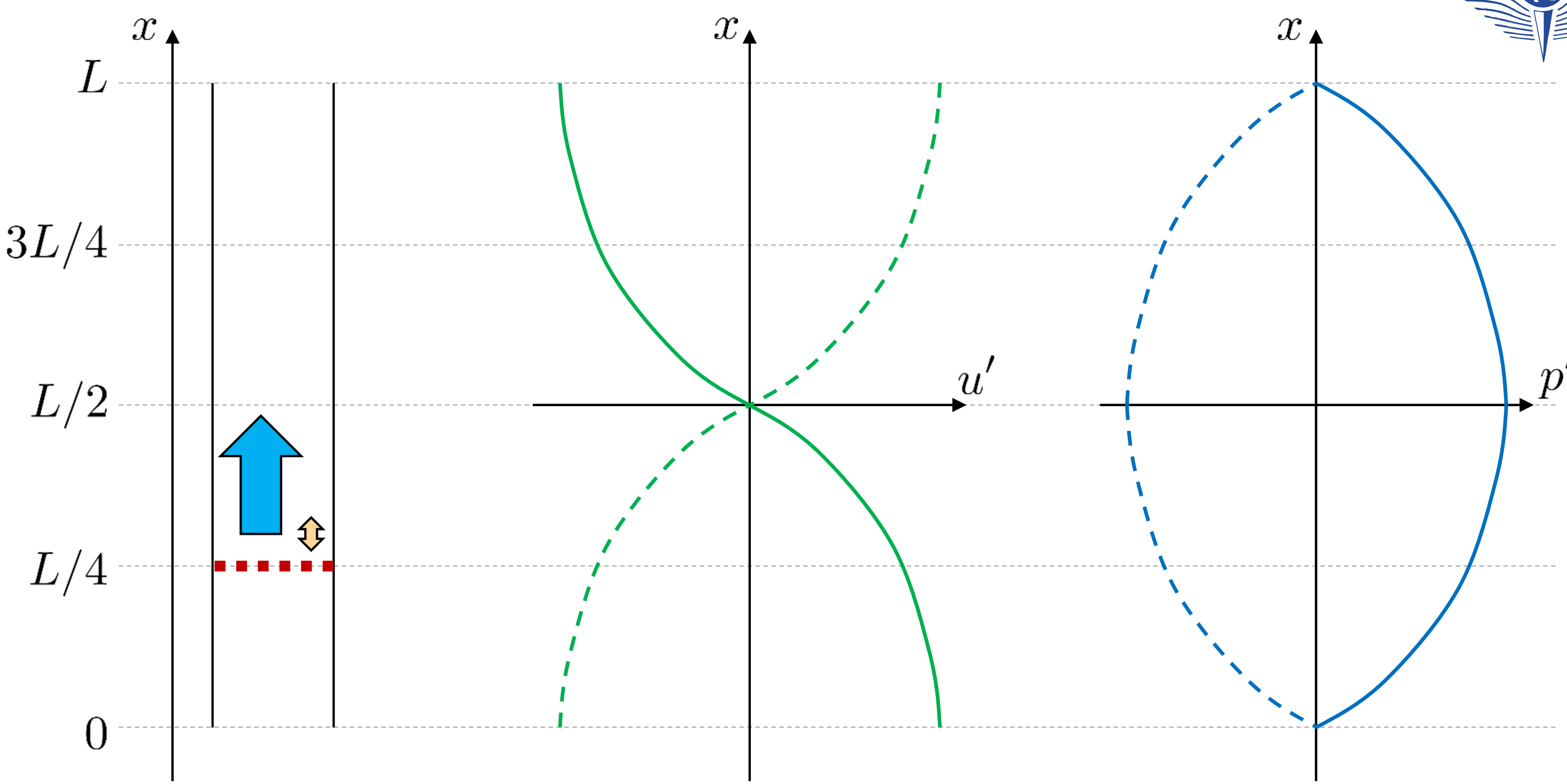
$$p' = p'_d + p'_r$$

$$|\mathbf{I}| \approx 4\,\frac{(p'_d)^2}{\rho_0 c_0}$$

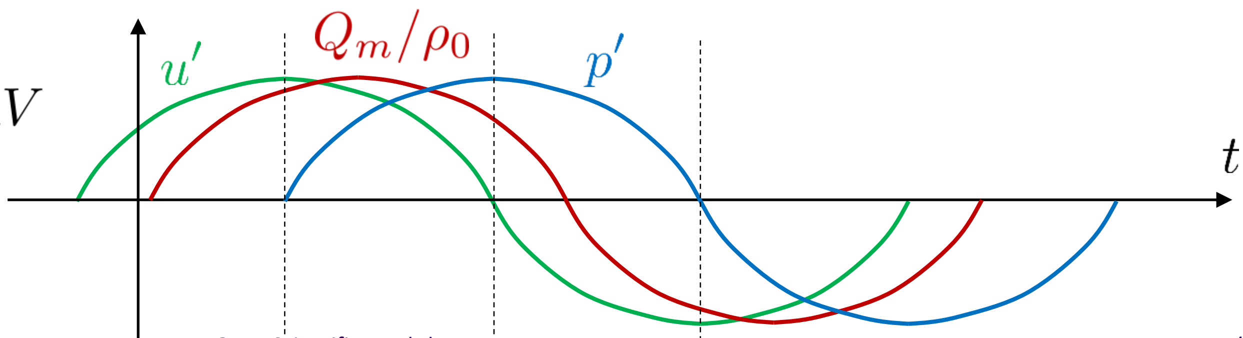
$$\langle P \rangle \approx \frac{1}{2}\left(4\pi r^2 |\mathbf{I}|\right) = 2\left(4\pi r^2\,\frac{(p'_d)^2}{\rho_0 c_0}\right) = 2\,\langle P_0 \rangle$$



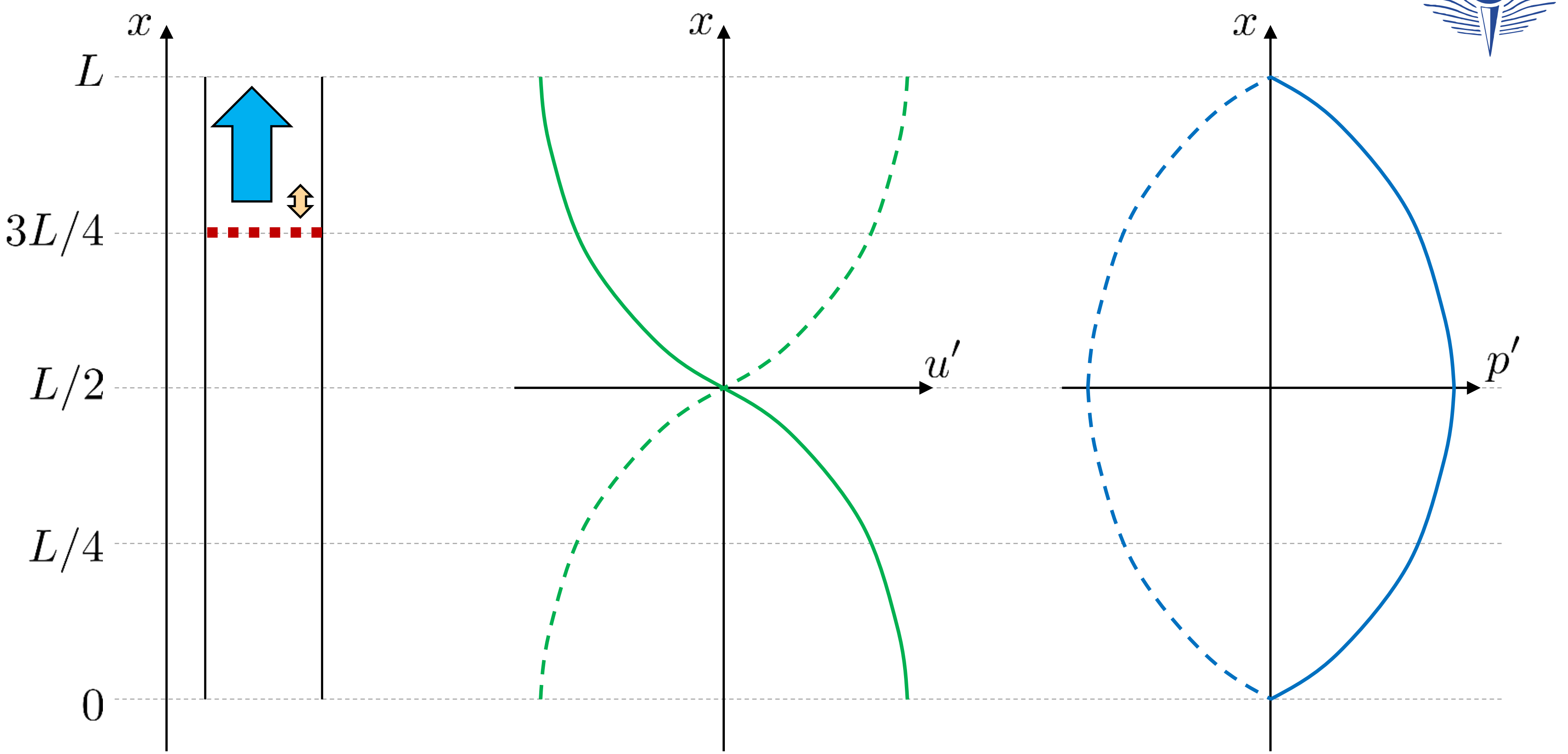
2nd example: Rijke tube



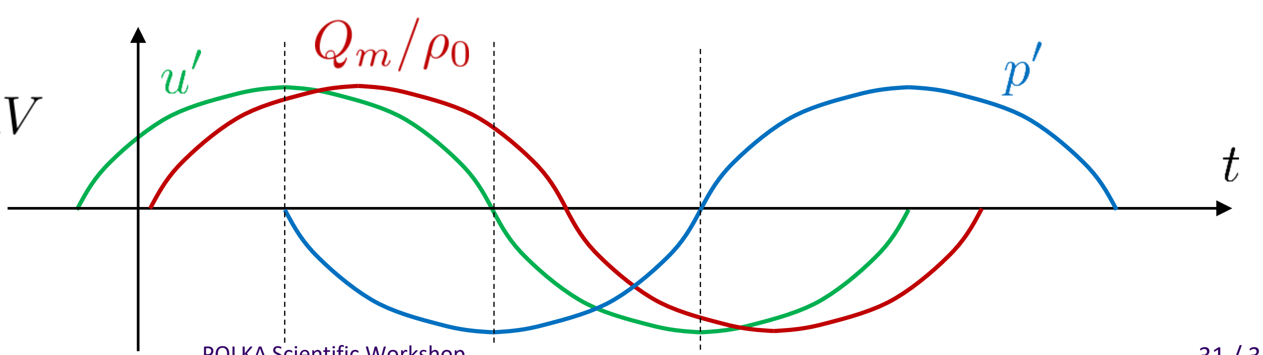
$$\langle P \rangle = \iiint_V \left\langle p' \, \frac{Q_m}{\rho_0} \right\rangle \mathrm{d}V$$



2nd example: Rijke tube



$$\langle P \rangle = \iiint_V \left\langle p' \, \frac{Q_m}{\rho_0} \right\rangle \mathrm{d}V$$



Summary

- Assuming small amplitude acoustic perturbations, the equations of fluid motion can be linearized and used to derive a wave equation for these perturbations.
- The relationship between the perturbations are given by the linearized momentum equation and the linearized constitutive equation:

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' \qquad\qquad p' = c_0^2 \rho' + \left(\frac{\partial p}{\partial s}\right)_\rho s'$$

- The sources of the acoustic field can be due to
  - Unsteady mass injection or entropy fluctuations → monopolar character.
  - Non-uniform forces → dipolar character.
  - Fluctuating viscous stresses or Reynolds stresses → quadrupolar character.
- Each of these sources has a different radiation efficiency in free field.
- Assuming a decoupling between the sound production and propagation, the analogies provide an explicit integral solution for the acoustical field at the listener position
  - Improves numerical robustness
  - Permits drawing scaling laws

A few references

- A. Pierce, *Acoustics: an Introduction to its Physical Principles and Applications*,
  McGraw-Hill Book Company Inc., New York, 1981.
- S.W. Rienstra and A. Hirschberg, *An Introduction To Acoustics* (corrections), Report IWDE 01-03 May 2001, revision every year or so…
- M.E. Goldstein, *Aeroacoustics*, McGraw-Hill International Book Company, 1976.
- A.P. Dowling and J.E. Ffowcs Williams, *Sound and Sources of Sound*, Ellis Horwood-Publishers, 1983.
- D.G. Crighton, A.P. Dowling, J.E. Ffowcs Williams, M. Heckl and F.G. Leppington, *Modern Methods in Analytical Acoustics*, Springer-Verlag London, 1992.

- And of course: the VKI Lecture Series…