Green's function methods in thermoacoustics

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Outline

1. The tailored Green's function
2. Solution of the acoustic analogy equation
3. Time history calculations
4. Stability analysis for individual modes
5. Stability analysis in the frequency domain
6. Outlook and potential extensions

1. The tailored Green's function

Green's function:

impulse source at point $\vec{x}^*, t^* \rightarrow$ response at point $\vec{x}, t$

governing equation: $$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \nabla^2 G = \delta(\vec{x} - \vec{x}^*)\delta(t - t^*)$$

Tailored Green's function:

also satisfies the boundary conditions (resonator)

General form: $$G(x, x^*, t - t^*) = \mathrm{H}(t - t^*)\sum_{n=1}^{\infty} G_n(x, x^*)\,\mathrm{e}^{-\mathrm{i}\omega_n (t - t^*)}$$

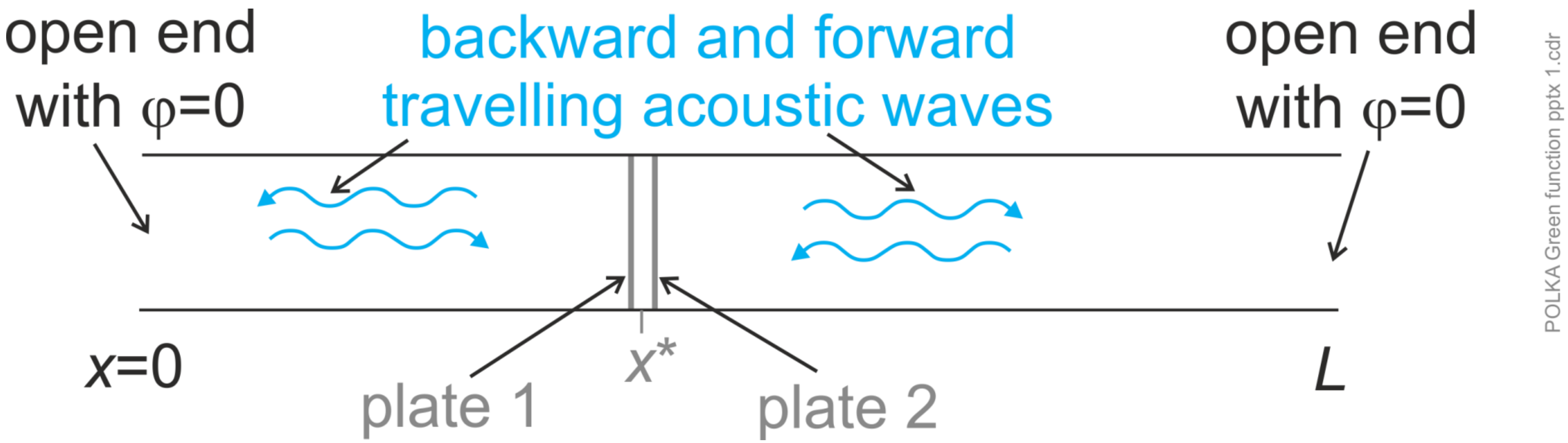
Superposition of modes n, where

H : Heaviside function (to ensure causality)

ω_n : eigenfrequencies of the resonator (with steady, but no unsteady, heat source)

G_n : Green's function amplitudes

Example: 1-D tube with open ends



φ: velocity potential

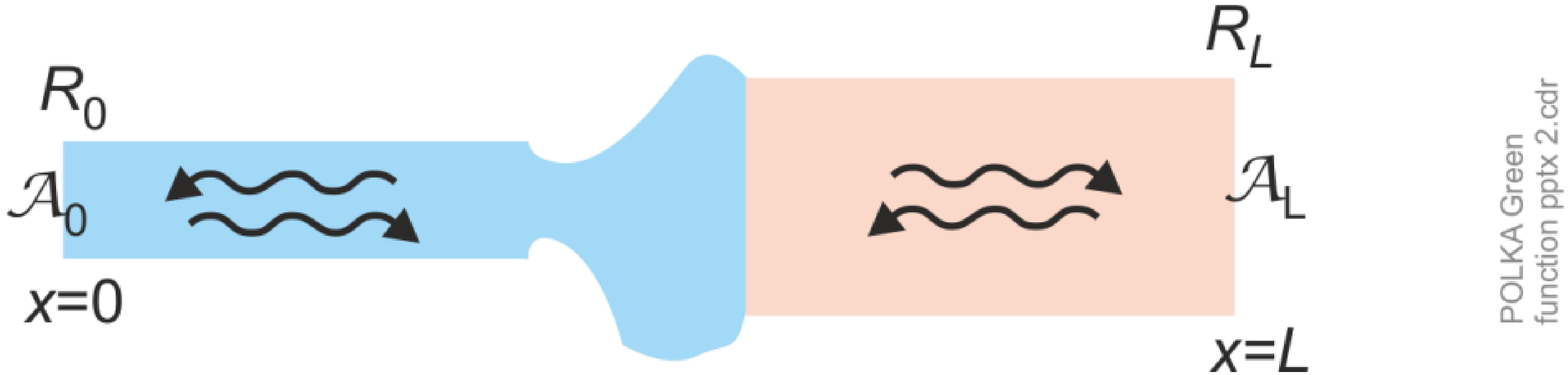
Results for Green's function:

$$\omega_n = \frac{n\pi c}{L}$$

$$G_n(x,x^*) = \begin{cases} \dfrac{(-1)^n}{n} \sin\dfrac{\omega_n x}{c} \sin\dfrac{\omega_n (x^* - L)}{c} & \text{for} \quad x < x^* \\[2ex] \dfrac{(-1)^n}{n} \sin\dfrac{\omega_n (x - L)}{c} \sin\dfrac{\omega_n x^*}{c} & \text{for} \quad x > x^* \end{cases}$$

ω_n and $G_n(x, x^*)$ can be calculated analytically for semi-1D configurations, with:

- jump in mean temperature
- ends described by reflection coefficients R_0 and R_L
- jump in cross-sectional area
- localised blockage



Summary

- The tailored Green's function is the response of a fluid with boundaries (typically a fluid within a resonator).
- tailored to the geometry of the resonator
- superposition of resonator modes
- can in principle be measured

2. Solution of the acoustic analogy equation

Analogy equation for the velocity potential φ:

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} = \underbrace{-\frac{\gamma - 1}{c^2} q'(x,t)}_{\text{forcing term}}$$

fluctuations of rate of heat release (per unit mass)

Forced PDE → suitable for Green's function approach

Governing equation for the Green's function:

$$\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} - \frac{\partial^2 G}{\partial x^2} = \delta(x - x^*)\delta(t - t^*)$$

Combine with equation for φ to get integral equation:

$\rightarrow$ integral equation:

$$\varphi(x,t) = -\frac{\gamma-1}{c^2}\int_{t^*=0}^{t} \underbrace{\int_{x^*=0}^{L} G(x,x^*,t-t^*)\, q'(x^*,t^*)\, \mathrm{d}x^*}_{} \, \mathrm{d}t^*$$

can be calculated for a compact heat source at x_q, $q'(x,t) = q(t)\,\delta(x-x_q)$

$$\varphi(x,t) = -\frac{\gamma-1}{c^2}\int_{t^*=0}^{t} G(x,x_q,t-t^*)\, q(t^*)\, \mathrm{d}t^* \quad \Bigg| \; \frac{\partial}{\partial x}, \quad \text{evaluate at } x = x_q$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=x_q} \qquad \left.\frac{\partial G}{\partial x}\right|_{\substack{x=x_q \\ x^*=x_q}}$$

equation for velocity at x_q:

$$u_q'(t) = -\frac{\gamma-1}{c^2}\int_{t^*=0}^{t} \left.\frac{\partial G}{\partial x}\right|_{\substack{x=x_q \\ x^*=x_q}} q(t^*)\, \mathrm{d}t^*$$

Assume heat release law of the form: $q(t) = \mathcal{F}(u_q(t))$
(linear or nonlinear)

Then

$$\underbrace{u_q(t)}_{\text{velocity at current time}} = -\frac{\gamma-1}{c^2}\int_{t^*=0}^{t} \left.\frac{\partial G}{\partial x}\right|_{\substack{x=x_q \\ x^*=x_q}} \mathcal{F}(\underbrace{u_q(t^*)}_{\text{velocity at earlier times}})\,\mathrm{d}t^*$$

This is a Volterra integral equation for $u_q(t)$.

Summary

- Acoustic analogy equation is PDE with forcing term
- Convert into integral equation with tailored Green's function
- For compact heat source, get Volterra integral equation for $u_q(t)$
- Can be solved for given initial conditions

3. Time history calculations

Solve integral equation with time-stepping method:

Discretise: $t \rightarrow t_m = 0, \Delta t, 2\Delta t, \ldots\ m\Delta t$

$$t^* \rightarrow t_j^* = 0, \Delta t, 2\Delta t, \ldots\ j\Delta t$$

$$\int_{t^*=0}^{t} \ldots \mathrm{d}t^* \rightarrow \sum_{j=1}^{m} \ldots \Delta t$$

Introduce abbreviation $g_n = -\dfrac{\gamma - 1}{c^2} \dfrac{\partial G_n}{\partial x}\bigg|_{\substack{x=x_q \\ x^*=x_q}}$, then

number of modes considered

$$u_q(t) = \int_{t^*=0}^{t} \sum_{n=1}^{N} g_n\, \mathrm{e}^{-\mathrm{i}\omega_n (t-t^*)} q(t^*) \mathrm{d}t^*$$

Collect terms with t^*:

$$u_q(t) = \sum_{n=1}^{N} g_n\, e^{-i\omega_n t} \underbrace{\int_{t^*=0}^{t} e^{i\omega_n t^*} q(t^*)\, dt^*}_{\mathcal{I}_n(t)\ \text{(abbreviation)}}$$

Split up integration range: $\displaystyle\int_{t^*=0}^{t} = \int_{t^*=0}^{t-\Delta t} + \int_{t^*=t-\Delta t}^{t}$

Then

$$\mathcal{I}_n(t) = \underbrace{\int_{t^*=0}^{t-\Delta t} e^{i\omega_n t^*} q(t^*)\, dt^*}_{=\mathcal{I}_n(t-\Delta t)} + \underbrace{\int_{t^*=t-\Delta t}^{t} e^{i\omega_n t^*} q(t^*)\, dt^*}_{\approx\, q(t) \underbrace{\int_{t^*=t-\Delta t}^{t} e^{i\omega_n t^*} dt^*}_{= \frac{e^{i\omega_n t}}{i\omega_n}\left(1-e^{-i\omega_n \Delta t}\right)}}$$

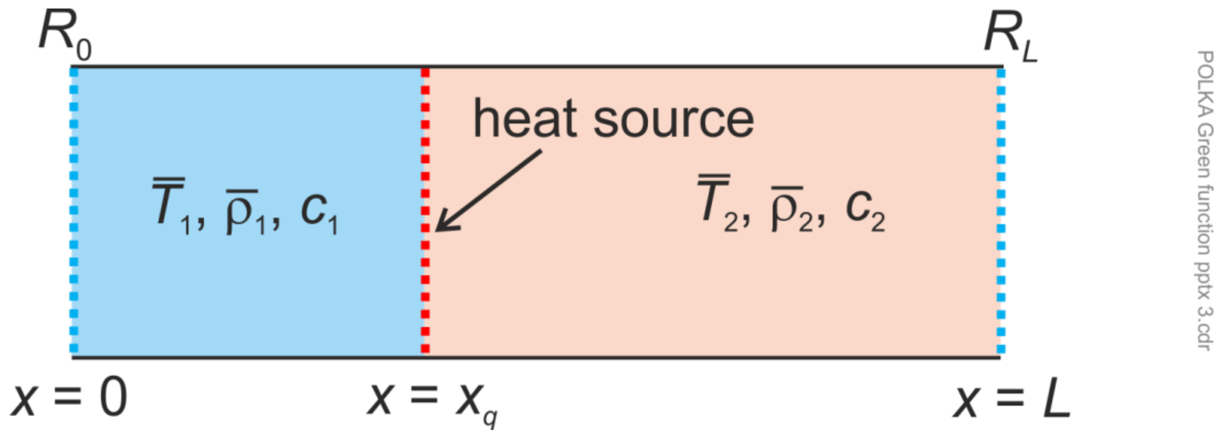
Iteration scheme $$u_q(t) = \sum_{n=1}^{N} g_n \, \mathrm{e}^{-\mathrm{i}\omega_n t} \, \mathcal{I}_n(t)$$

with $$\mathcal{I}_n(t) = \mathcal{I}_n(t-\Delta t) + q(t) \frac{\mathrm{e}^{\mathrm{i}\omega_n t}}{\mathrm{i}\omega_n} \left(1 - \mathrm{e}^{-\mathrm{i}\omega_n \Delta t}\right)$$

$$q(t) = \mathcal{F}(u_q(t))$$

Only N terms need to be calculated in each iteration step.

Example: Nonlinear interaction between two modes



$$G(x,x^*,t-t^*) = \mathrm{H}(t-t^*)\left[\underbrace{G_1(x,x^*)\,\mathrm{e}^{-\mathrm{i}\omega_1(t-t^*)}}_{\text{resonator mode 1}} + \underbrace{G_2(x,x^*)\,\mathrm{e}^{-\mathrm{i}\omega_2(t-t^*)}}_{\text{resonator mode 2}}\right]$$

$$q(t) = K\left[n_1 \int_{\tau=0}^{\infty} u_q(t-\tau) D_1(\tau)\,\mathrm{d}\tau - n_2 \int_{\tau=0}^{\infty} u_q(t-\tau) D_2(\tau)\,\mathrm{d}\tau\right]$$

distribution around time-lag τ_1

distribution around time-lag τ_2

with $K = \dfrac{\bar{Q}}{S\bar{u}_2\bar{\rho}_2}$

Alessandra Bigongiari and Maria Heckl (2018) Analysis of the interaction of thermoacoustic modes with a Green's function approach. *International Journal of Spray and Combustion Dynamics*, Vol. 10(4), pp. 326-336

Time-lag distributions around τ_1 and τ_2 :

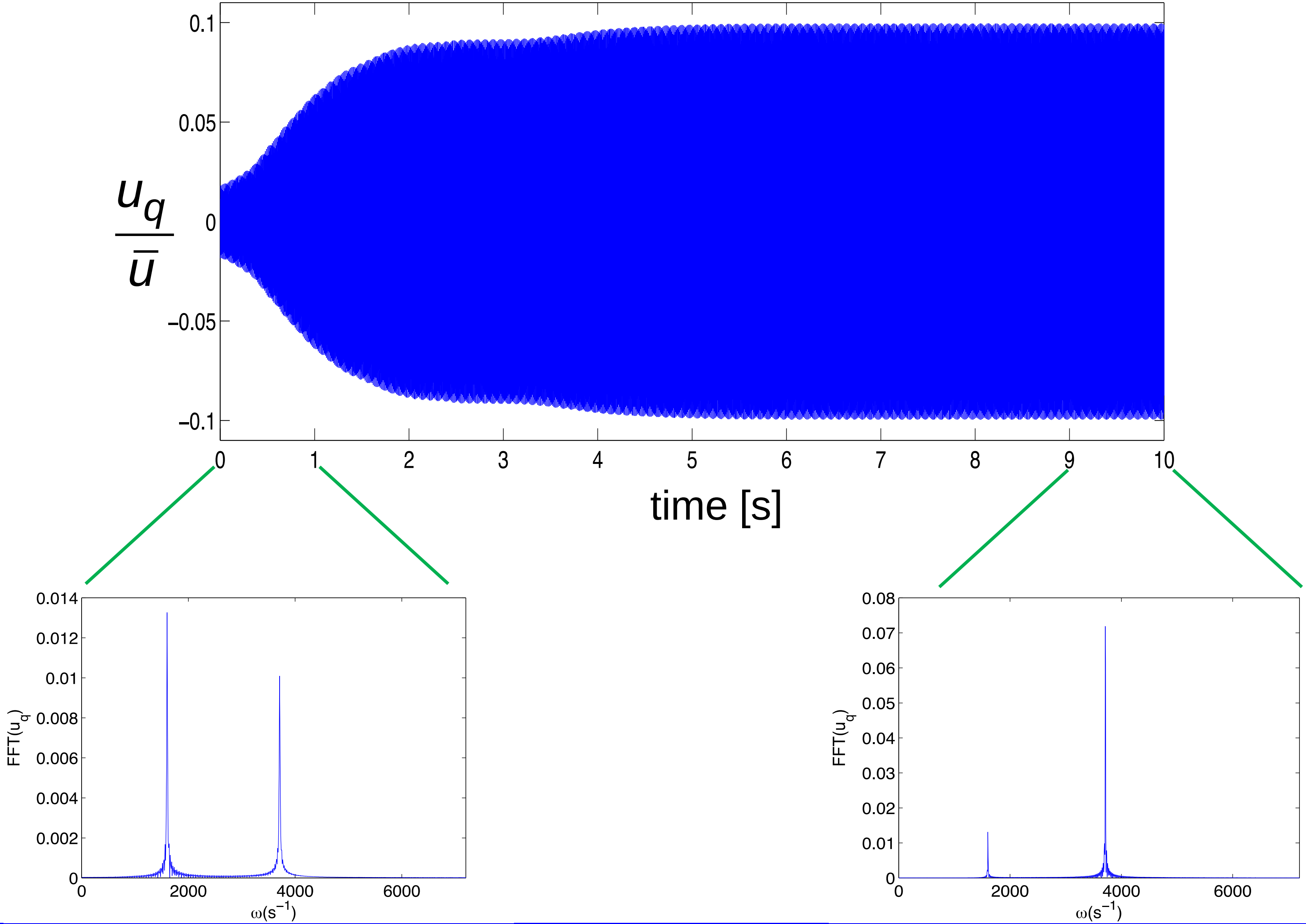
$$D_1(\tau) = \frac{2}{\sigma_1\sqrt{2\pi}} e^{-\frac{(\tau-\tau_1)^2}{2\sigma_1^2}}$$

$$D_2(\tau) = \frac{2}{\sigma_2\sqrt{2\pi}} e^{-\frac{(\tau-\tau_2)^2}{2\sigma_2^2}}$$

The time-lags are amplitude-dependent:

$$\left.\begin{aligned}\tau_1 &= 4.1 - 6.6\frac{a}{\bar{u}}\,[10^{-3}\,\mathrm{s}]\\ \tau_2 &= 6.3 - 1.9\frac{a}{\bar{u}}\,[10^{-3}\,\mathrm{s}]\end{aligned}\right\}$$ from combustion CFD simulations

Results



$\frac{u_q}{\bar{u}}$
0.1
0.05
0
−0.05
−0.1
0
1
2
3
4
5
6
7
8
9
10
time [s]
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
FFT(u_q)
0
2000
4000
6000
ω(s^{-1})
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
FFT(u_q)
0
2000
4000
6000
ω(s^{-1})

Summary

- explicit calculation, stepping forward in time.

- no increase in numerical effort as iteration progresses

- works for *any* heat release law of the form $q(t) = \mathcal{F}(u_q(t))$

- resulting time history gives information on
  frequency
  growth rate
  limit cycle amplitude
  ...

4. Stability analysis for individual modes

Consider single mode *n*

→ integral equation for the velocity:

$$u_n(t) = \int_{t^*=0}^{t} g_n \, e^{-i\omega_n (t-t^*)} q(t^*) \mathrm{d}t^*$$

Convert to differential equation for velocity of mode *n*:

$$\underbrace{\ddot{u}_n - 2\mathrm{Im}(\omega_n)\,\dot{u}_n + |\omega_n|^2 u_n}_{\text{damped harmonic oscillator}} = \underbrace{-\mathrm{Im}(\omega_n g_n^*)\, q(t) + \mathrm{Re}(g_n)\,\dot{q}(t)}_{\text{forcing term}}$$

Assume heat release law $q(t) = [n_1 u(t-\tau) - n_0 u(t)]$

amplitude-dependent coefficients
can be obtained e.g. from FDF measurements

Heckl, M.A. (2013) Analytical model of nonlinear thermo-acoustic effects in a matrix burner. *Journal of Sound and Vibration* 332, 4021-4036.

Substitute into oscillator equation:

$$\ddot{u}_n + \underbrace{\left[-2\mathrm{Im}(\omega_n) + n_0 \mathrm{Re}(g_n)\right]}_{=c_1}\dot{u}_n + \underbrace{\left[|\omega_n|^2 - n_0 \mathrm{Im}(\omega_n g_n^*)\right]}_{=c_0} u_n =$$

$$= \underbrace{\left[-n_1 \mathrm{Im}(\omega_n g_n^*)\right]}_{=b_0} u_n(t-\tau) + \underbrace{\left[n_1 \mathrm{Re}(g_n)\right]}_{=b_1} \dot{u}_n(t-\tau)$$

We look for steady limit cycle solutions: $u_n(t) = a\cos \underbrace{\Omega}_{\approx \omega_n} t$

For *any* time-lag:

$$u_n(t-\tau) = a\cos\Omega(t-\tau) = (\cos\Omega\tau)\, u_n(t) - \frac{\sin\Omega\tau}{\Omega}\dot{u}_n(t)$$

[redacted]

ODE for $u_n(t)$:

$$\ddot{u}_n(t) + \underbrace{\left[c_1 + b_0 \frac{\sin\omega_n\tau}{\omega_n} - b_1\cos\omega_n\tau\right]}_{=a_1} \dot{u}_n(t) + \underbrace{\left[c_0 - b_0\cos\omega_n\tau - b_1\omega_n\sin\omega_n\tau\right]}_{=a_0} u_n(t) = 0$$

a_0 : oscillation frequency (squared)

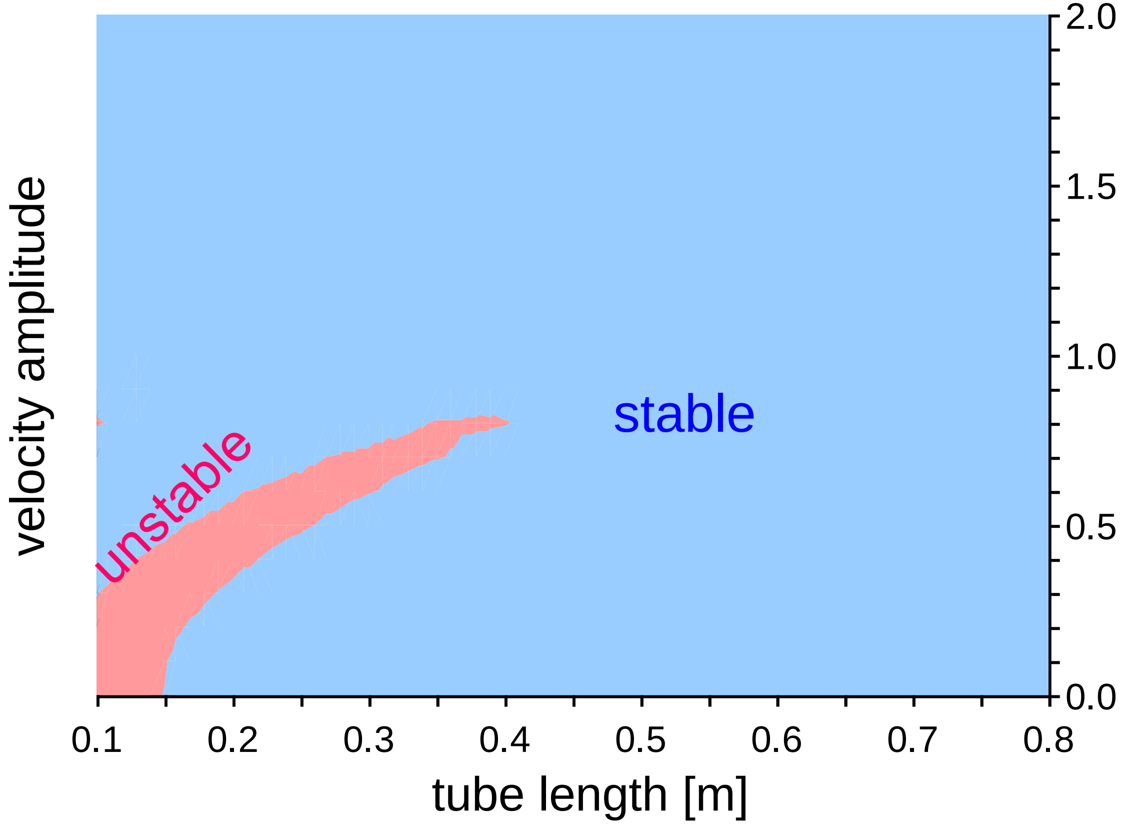
a_1 : damping coefficient, amplitude-dependent

$a_1 > 0$: stability, $a_1 < 0$: instability

The stability behaviour can be examined at different amplitudes for various system parameters.

Example: λ/4 resonator with variable length and amplitude-dependent time-lag law

time-lag: $\tau = \tau_0 + \tau_1 (a / \bar{u})^2$, a: velocity amplitude



Summary

- Single mode in isolation behaves like harmonic oscillator, forced by heat release rate
- For heat release rate with time-lags, mode behaves like damped oscillator
- Damping coefficient gives stability behaviour
- Stability depends on Green's function amplitude and frequency of mode, also on time-lags

5. Stability analysis in the frequency domain

Observation: instability frequencies are discrete

→ Assumption: superposition of thermoacoustic modes

$$u_q(t) = \mathrm{Re} \sum_{m=1}^{\infty} u_m \mathrm{e}^{-\mathrm{i}\Omega_m t}$$

velocity at the heat source

complex velocity amplitude of t-a mode *m*

complex frequency of t-a mode *m*

$\mathrm{Im}(\Omega_m)$ gives the stability behaviour of t-a mode *m*

compare with Green's function:

$$G(x,x^*,t-t^*) = \mathrm{H}(t-t^*) \sum_{n=1}^{\infty} G_n(x,x^*) \mathrm{e}^{-\mathrm{i}\omega_n (t-t^*)}$$

complex eigenfrequency of resonator mode *n*

$\mathrm{Im}(\omega_n) < 0$, i.e. never unstable

Aim: determine Ω_m and u_m

Requires several mathematical steps: Volterra equation

Laplace transform, ...

equation for Ω_m :

$$(n_1 e^{i\Omega_m \tau} - n_0)\sum_{n=1}^{\infty}\left[\frac{g_n}{i(\omega_n - \Omega_m)} - \frac{g_n{}^*}{i(\omega_n{}^* + \Omega_m)}\right] = -\frac{2c^2\bar{u}S\bar{\rho}}{\bar{Q}(\gamma-1)}$$

where:

$$q'(x,t) = q(t)\delta(x - x_q)$$

$$q(t) = \frac{\bar{Q}}{S\bar{\rho}}\left[n_1 \frac{u_q(t-\tau)}{\bar{u}} - n_0 \frac{u_q(t)}{\bar{u}}\right]$$

$$g_n = \frac{\partial G_n(x,x^*)}{\partial x}\Bigg|_{\substack{x=x_q \\ x^*=x_q}}$$

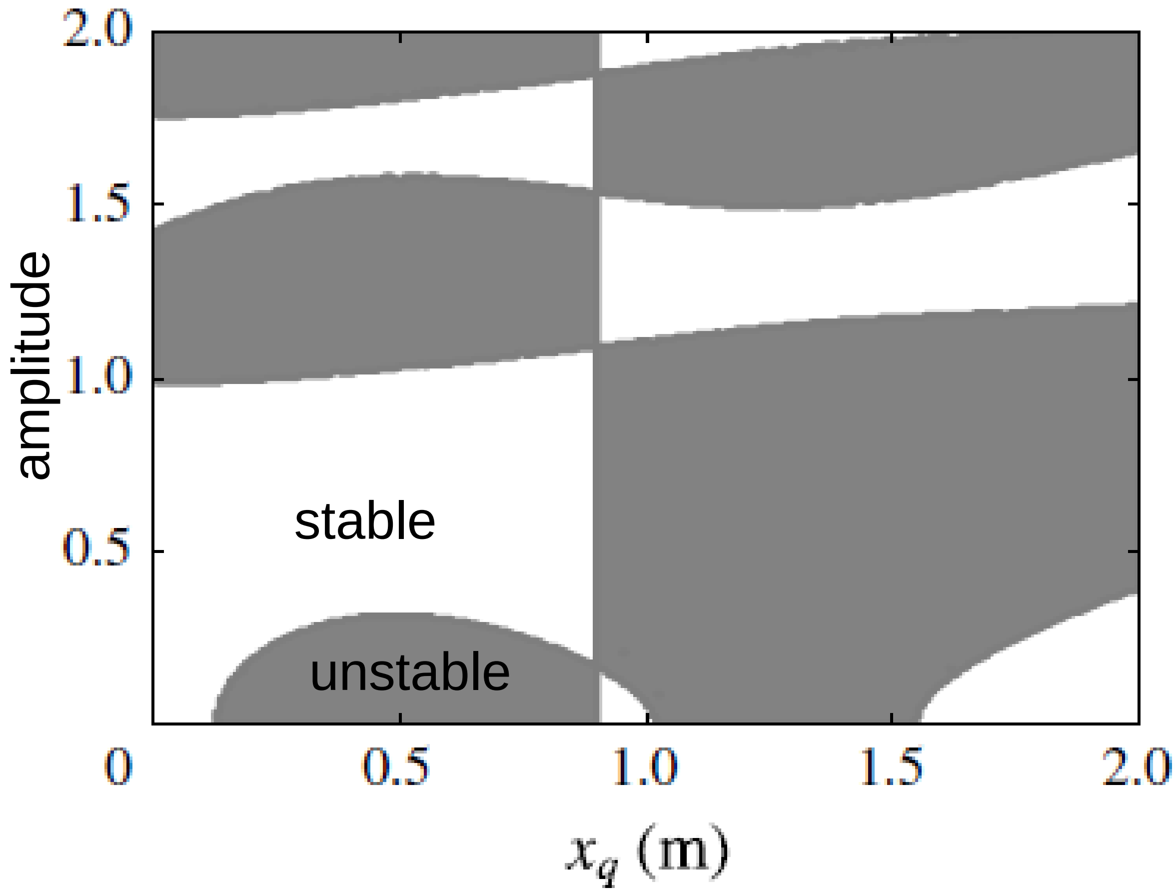
linear equations for u_m :

$$\begin{bmatrix} \text{matrix elements} \\ \text{depend on} \\ g_n, \omega_n, \Omega_m, n_0, n_1, \tau \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \text{vector elements} \\ \text{depend on} \\ \text{initial conditions} \end{bmatrix}$$

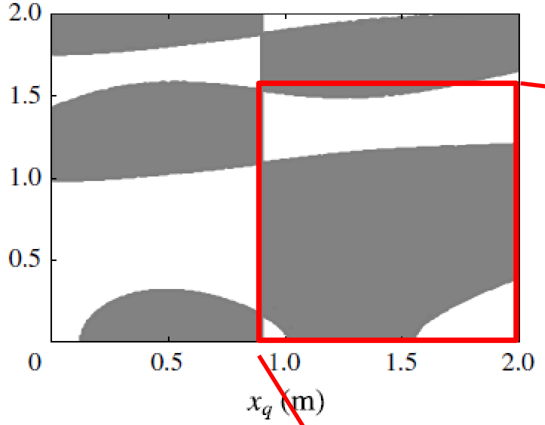
Stability predictions for amplitude-dependent time-lag

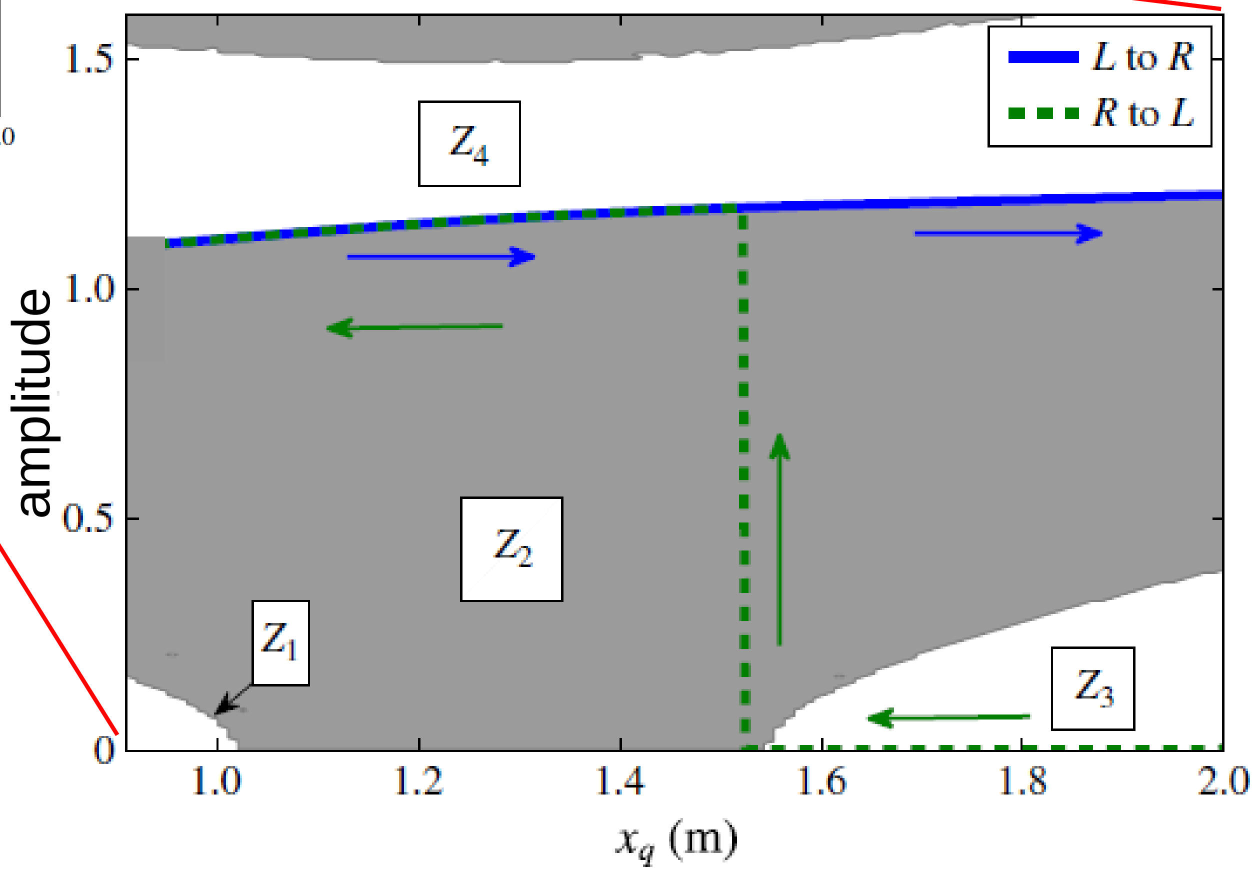
resonator: tube with open ends

time-lag : $\tau = 5\times10^{-3} + 4.4\times10^{-3}(a/\bar{u})^2$ [s]



Hysteresis behaviour





Bigongiari, A. & Heckl, M.A. (2016) A Green's function approach to the rapid prediction of thermoacoustic instabilities in combustors. *Journal of Fluid Mechanics* 798, 970-996.

Summary

- calculation faster than iterative solution of Volterra equation
- amplitude-dependence can be incorporated
- stability maps explain potential hysteresis behaviour
- to be used with caution for multiple modes

6. Outlook and potential extensions

POLKA

- Add more modes
- Investigate interaction between modes
- Use more realistic heat release law
- …

Collaboration with Beihang University (Xiaoyu Wang)

3-D Green's function, cartesian coordinates:

→ model flame in a box

3-D Green's function, cylinder coordinates:

→ model flames in an annular combustion chamber

Thank you!

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