Determining thermo-acoustic stability of a system whose boundary conditions are represented by strictly positive real transfer functions.

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Abstract

The ultimate goal of the present research is to establish a methodology using which one can characterize the thermo-acoustic quality (figure of merit) of a given burner with flame. For this purpose, the probability of a certain burner/flame to be in either a stable or unstable regime when it is embedded in a randomly selected acoustic environment (similar to a combustor appliance) should be evaluated. An approach presented in this contribution consists of performing multiple calculations for the (in)-stability of a system composed of a known burner with flame and acoustically passive arbitrary upstream and downstream reflection coefficients. In this paper, a low order analytical network model of the acoustic system is used. Properties of strictly positive real functions are used to model the random frequency dependence of passive reflections. The implementation and testing of this particular method to generate random, frequency dependent acoustic embedding for the burner is the core subject of the present contribution. Within this method, initially, the roots of a Hurwitz polynomial are randomly selected and this polynomial is taken as the denominator of the impedance function, subsequently, the corresponding numerator polynomial coefficients are computed to obtain an impedance function that is strictly positive real. Then, this function is transformed to represent a reflection coefficient function in complex variable s and is used as an embedding to evaluate the given burner's flame stability by calculating the system's complex eigen frequencies for various upstream and downstream reflection coefficients.

Introduction

In domestic and industrial combustion devices, thermo-acoustic instability caused by coupling between the flame and system acoustics is a major threat as it mitigates the safe operating margin of an appliance and leads to a decrease in its efficiency. Oscillations in the flow field result in fluctuating heat release of the burner which produces sound and the resulting acoustic wave travels to system boundaries and gets reflected back to the flame forming a closed feedback loop. Therefore, to determine the stability of a system as shown in Figure 1, it becomes imperative to evaluate the stability of a burner/flame combined with upstream and downstream boundary conditions.

Combustion system dynamics have been studied experimentally, computationally and analytically using the acoustic network framework in the last few years. The advantage of analytical study comes from its simplicity to describe a system, ease of application, low cost and accuracy. Generally, a thermo-acoustic network system modelling requires knowledge about a) the flame response to acoustics and b) acoustics feedback at the boundaries in the form of two-port network elements. Such a model incorporates various elements of a system in terms of their transfer matrices to generate a global transfer matrix which has two acoustic inputs and two acoustic outputs (often given in terms of Riemann invariants, f and g) [1].

Multiple studies have shown that burner/flame stability is determined by the system acoustics, however recent studies have also shown that in the absence of sound reflection, a burner flame may be intrinsically unstable [2].

In [3], the authors have tried to find a method to evaluate the figure of merit for different burners based on their stability with randomly varying upstream and downstream frequency independent terminations. They have also proposed an idea that frequency dependent terminations in the form of strictly positive real function impedances could be used to determine burner's figure of merit.

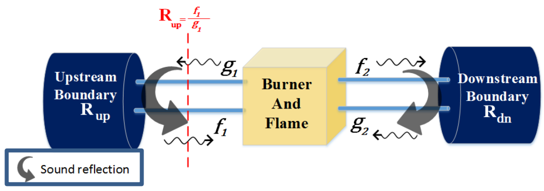

Figure 1: Thermo-acoustic system

Based on this suggestion, in this paper we have developed a methodology where the upstream and downstream boundary conditions are represented by passive impedances in terms of strictly positive real functions and the burner's flame response is given by the flame Transfer Function. This ensures the flame as a unique active element in the system, while the upstream and downstream terminations are passive elements. The complete system is then modelled using a two-port network model and system stability is determined. The upstream and downstream travelling acoustic waves are described as:

$$f = \frac{1}{2}(v' + \frac{p'}{\rho_0 c})$$

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$$g = \frac{1}{2}(v' - \frac{p'}{\rho_0 c})$$ (1)

Where, p' and v' are acoustic pressure and velocity perturbations respectively. And $\rho_0 c$ is the specific impedance of the gas.

In circuit theory, the notion of passivity is attributed to a system which consumes energy but doesn't generate energy [4]. Any function which satisfies the properties of a strictly positive real (SPR) function can then be used as driving point impedance of a passive element and conversely, a passive element has a driving point impedance which is SPR. Hence, in this paper, we assume upstream and downstream terminations impedances as passive elements which are represented/modelled by SPR transfer functions. The method to automatically generate such a SPR function is adopted from literature and is presented in the next section [6]. It is important to remember that the network itself isn't passive due to the presence of an active flame response.

Reflection Coefficient from Strictly Positive Real Function Impedance

The first step is to automatically generate a strictly positive real (SPR) transfer function as an impedance for a boundary termination.

The poles and zeros of such a function are located in the left half plane making it stable.

A rational function $H(s)$ of the complex variable $s = \sigma + j\omega$ is SPR if: [5]

a. $H(s)$ is analytic in $Re[s] \geq 0$
b. $Re[H(j\omega)] > \varepsilon \qquad \forall\omega \in [-\infty, \infty]$

Where ε is a positive constant.

In this paper, we have used the methodology from [6] where authors propose an algorithm where, if we have a known Hurwitz polynomial $q(.)$, the algorithm provides a polynomial $p(.)$ such that $H(.) = p(.)/q(.)$ is SPR.

In [6] the following definitions and theorems for SPR functions are given:

Definition 1: Let $H(s)$ be a rational function of form $H(s) = p(s)/q(s)$ where $p(s) \in P^m$ and $q(s) \in P^n$ then $H(s)$ is SPR if it satisfies above mentioned (a) and (b).

Theorem 1[7]: Assuming q and p are polynomials of same order, $H(s)$ is SPR if and only if

a) $H(0) > 0$;
b) $p(s)$ and $q(s)$ are Hurwitz polynomials;
c) $q(s) + j\alpha p(s)$ is Hurwitz for all $\alpha \in \Re^+$

To determine the polynomial $p(s)$, we start by considering a polynomial $q(s)$ which needs to be Hurwitz. By definition of Hurwitz polynomial [8]:

Definition 2: Let

$$\delta(s) = \delta_0 + \delta_1 s + \cdots + \delta_n s^n$$

be a given real polynomial of degree n, then it can be written as

$$\delta(s) = \delta_e(s^2) + s\delta_o(s^2)$$

where $\delta_e(s^2)$, $s\delta_o(s^2)$ are made up of terms with even and odd powers of s respectively.

It must also satisfy Hermite Biehler Theorem which states that any polynomial of the form $\delta(s)$ is Hurwitz stable if and only if all the zeroes of $\delta_e(-\omega^2), \delta_o(-\omega^2)$ are real and distinct, δ_n and δ_{n-1} are of the same sign and the non-negative real zeros satisfy the following interlacing property:

$$0 < \omega_{e1} < \omega_{o1} < \omega_{e2} < \omega_{o2} < \omega_{e3} \ldots$$ (2)

Such a Hurwitz polynomial should have monotonic increasing phase, that is, the phase of $\delta(j\omega)$ is a continuous and strictly increasing function of $\omega(-\infty, \infty)$.

As mentioned in [6], we begin our search to find $p(s)$ by assuming roots of $q_e(s)$ and $q_o(s)$ which are interlaced and obtain even and odd parts of $q(s) = q_e(s) + q_o(s)$. Let us define $g(.)$ and $h(.)$ such that

$$g(s^2) = q_e(s) = q_0 + q_2 s^2 + q_4 s^4 + \cdots$$
$$h(s^2) = q_o(s)/s = q_1 + q_3 s^2 + q_5 s^4 + \cdots$$
and, $q(s) = g(s^2) + sh(s^2)$

Once all the above properties are satisfied by a polynomial, we can use it as the denominator $q(s)$ to obtain $H(s)$. To find the numerator $p(s)$, it must also be divided in terms of its odd and even part as follows:

$$u(s^2) = p_e(s) = p_0 + p_2 s^2 + p_4 s^4 + \cdots$$
$$v(s^2) = p_o(s)/s = p_1 + p_3 s^2 + p_5 s^4 + \cdots$$

such that,

$$p(s) = u(s^2) + sv(s^2)$$

The roots of $u(-\omega^2)$ and $v(-\omega^2)$ must be chosen from $g(-\omega^2)$ and $h(-\omega^2)$ in the following manner (when $s = j\omega$):

$$\omega_{gi} < \omega_{ui} < \omega_{hi}$$
$$\omega_{hi} < \omega_{vi} < \omega_{g(i+1)}$$ (3)

$$u(-\omega^2) = K(\omega_{u1}^2 - \omega^2)(\omega_{u2}^2 - \omega^2) \ldots$$
$$= KU(-\omega^2), \qquad K \in \Re^+$$
$$v(-\omega^2) = (\omega_{v1}^2 - \omega^2)(\omega_{v2}^2 - \omega^2) \ldots$$ (4)

In [6], the authors mention 2 algorithms for obtaining $p(s)$ from above mentioned equations. Both algorithms were tested and it was found that the second algorithm was faster and more simplified than the first one, as mentioned in [6] and therefore it was used in our paper as well.

In this algorithm, roots of $U(-\omega^2)$ and $v(-\omega^2)$ were chosen randomly such that they satisfy Equation (3), then an arbitrary value of K was specified and the resulting polynomial p/q was checked to be SPR. If not, then another value of K is chosen, and in this way, by iterating and sweeping values of K, multiple SPR functions were obtained. Below is an example for the SPR function obtained by following the above steps:

Example:

Let the roots of $q_e(s) = [1,3,5]$ and $q_o(s) = [2,4]$

Then we get

$$q(s) = 1s^6 + 1s^5 + 9s^4 + 6s^3 + 23s^2 + 8s + 15$$

Figure 2 and Figure 3 show interlacing zeros of even and odd polynomials and resulting monotonous increasing phase of $q(s)$.

Therefore, the roots for $\omega_{gi} = [1, 1.732, 2.236]$ and $\omega_{hi} = [1.414, 2]$

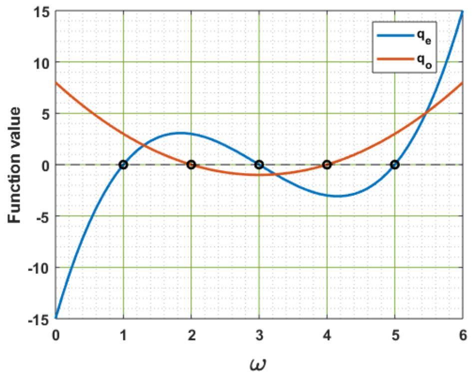


Figure 2: Interlacing zeros of even and odd polynomials of $q(s)$

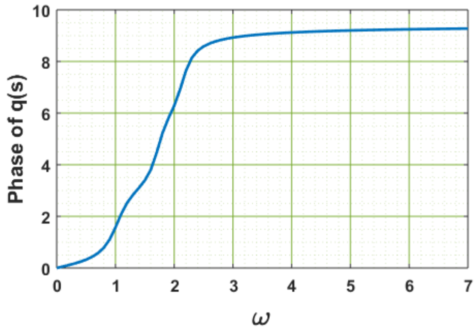


Figure 3: Monotonous increasing phase of chosen $q(s)$

Using above Equation (3) the roots of $U(-\omega^2)$ are obtained as [1.0062 1.7546 2.3175] and roots of $v(-\omega^2)$ as [1.4224 2.006]. For these set of values, any $0.001 \leq K \leq 25.75$ ($K_{min} \leq K \leq K_{max}$) will result in p/q to be a SPR function. Calculated $p(s)$ is also Hurwitz polynomial as it has interlacing zeros of even and odd powers of s and has a monotonously increasing phase.

Once we obtain a SPR function (as impedance $H(s) = Z$), we can transform it into Reflection Coefficient (RC) by the following equation:

$$RC(s) = \frac{H(s) - 1}{H(s) + 1} = \frac{p(s) - q(s)}{p(s) + q(s)}$$

Figure 4 and Figure 5 represents magnitude and phase of $RC(s)$ (polynomial of order ~ 6/6) for the set of $q(s)$ roots from the above mentioned example with different K values. The trend of $RC(s)$ magnitude variation always remains the same. As observed in Figure 4 and Figure 5, when we increase $K = K_{min}$ to some $K = K_{mid}$, the $RC(s)$ magnitude decreases from 1 to 0, and for any further increase in K from K_{mid} to K_{max}, the magnitude increases from 0 back to 1. Though we are calling this middle value, K_{mid}, it is not the midpoint between the two extremes of K, i.e. $(K_{max} - K_{min})/2 \neq K_{mid}$.

The goal to represent a termination in terms of a SPR transfer function impedance is achieved. This method provides us with the freedom to choose polynomial of any order to obtain our transfer function impedance, we can randomly select the roots of the denominator $q(s)$ and obtain a range of SPR functions by K variation. $RC(s)$ can be used to represent upstream and downstream terminations.

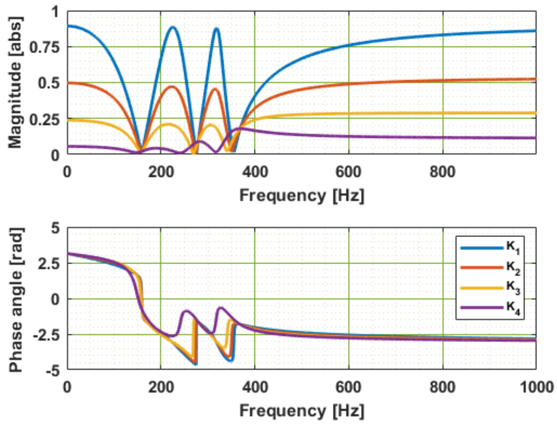


Figure 4: $RC(s)$ with 4 different values of K ($K_{min} < K_1 < K_2 < K_3 < K_4$)

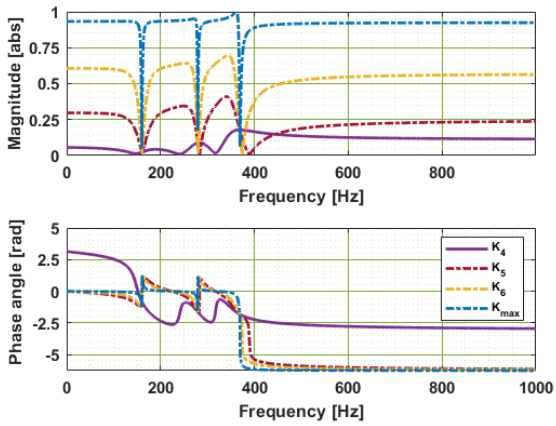


Figure 5: $RC(s)$ with 4 different values of K ($K_{min} < K_4 < K_5 < K_6 < K_{max}$)

Calculation of probability of instability as measure of the burner figure of merit.

The described method to generate a passive acoustic embedding can be used to evaluate the thermo-acoustic quality factor of the specified burner/flame. For this purpose, the burner, described by its transfer matrix, should be terminated by multiple passive terminations and for each of the embedding cases, the complex eigen frequencies should be evaluated. The probability to encounter an unstable operation may then serve as the thermo-acoustic figure of merit of the given flame. Below we follow this idea and show an example of the implementation of the proposed research program. The systematic application of the developed methodology is the subject of future research.

Flame transfer function and flame properties

In the linear limit, the response of a flame of a burner to acoustic oscillations can be represented by flame Transfer Function (TF). In the case of an acoustic velocity sensitive flame, the TF is often defined as

$$G(f) = \frac{Q'(f)/Q_0}{u'(f)/U_0},$$

where,

$Q'(f)$: unsteady heat release as a function of frequency

$u'(f)$: acoustic oscillations as a function of frequency

Q_0: mean heat release, and

U_0: mean velocity

This TF can be obtained experimentally or via CFD. In this paper, for the purpose of demonstrating the method to calculate the figure of merit of a burner, we use an analytical expression for a TF represented by the time delayed second order system with overshoot and damping factor to mimic a TF having similar characteristics as typically observed for an experimentally obtained TF, [3]

$$G(s) = \frac{1}{(A^2s^2 + 2\xi As + 1)} \cdot \exp(-s \cdot \tau_0), \tag{5}$$

$$A = \frac{1}{2\pi F_m}\sqrt{1-\xi^2}.$$

F_m: Frequency for maximum overshoot

ξ: Damping factor

Such a flame transfer function can then be plotted in terms of its gain and phase as shown in Figure 6(a). We have fixed the parameters: $\tau_0 = 0.025$ sec, $F_m = 100$ Hz and $\xi = 0.55$.

Network Model and Scattering Matrix

Within the limit of so-called "compact flames", when the flame/burner size $\delta \ll \lambda$, where λ is the wavelength of the acoustic wave, the burner can be treated as an acoustically compact lumped element. Accordingly, the burner with flame can be represented in terms of its Transfer Matrix (TM) [2]. Using linearized momentum equation and jump condition across the flame, the link between the flame TF and TM becomes [2]:

$$T(s) = \frac{1}{2}\begin{pmatrix} \varepsilon + 1 + \theta G(s) & \varepsilon - 1 - \theta G(s) \\ \varepsilon - 1 - \theta G(s) & \varepsilon + 1 + \theta G(s) \end{pmatrix},$$

where,

$\varepsilon = \frac{\rho_c c_c}{\rho_h c_h}$ is the ratio of specific impedances in the cold side and hot sides of the flame and $\theta = \frac{T_h}{T_c} - 1$ is the ratio of temperatures on the hot and cold sides. Combining the TM and reflections at the boundaries (the upstream and downstream side) of the burner, the relations between travelling waves (f and g) form the following system of equations:

$$f_1 = RC_{up}(s) \cdot g_1$$
$$f_2 = T(s)_{11} \cdot f_1 + T(s)_{12} \cdot g_1$$
$$g_2 = T(s)_{21} \cdot f_1 + T(s)_{22} \cdot g_1$$
$$g_2 = RC_{dn}(s) \cdot f_2.$$

Or, in matrix form:

$$\begin{bmatrix} 1 & -RC_{up} & 0 & 0 \\ 0 & 0 & -RC_{dn} & 1 \\ T_{11} & T_{12} & -1 & 0 \\ T_{21} & T_{22} & 0 & -1 \end{bmatrix}\begin{bmatrix} f_1 \\ g_1 \\ f_2 \\ g_2 \end{bmatrix} = 0. \tag{6}$$

Equation (6) will have non-trivial solution for f and g when the determinant of the first system matrix becomes 0. Hence, the dispersion relation will take the following form

$$T_{22} - RC_{dn}T_{12} + RC_{up}T_{21} - RC_{dn}RC_{dn}T_{22} = 0 \tag{7}$$

In this equation all terms may depend on the complex frequency s. By solving the Equation (7), we obtain the eigen frequencies $s_{eigen} = \sigma + j\omega$, where σ is the growth rate (rad/sec) and ω is the angular eigen frequency of the system. Positive (negative) value of σ represents an unstable (stable) dynamics.

Demonstration of Eigen frequency and growth rate calculations and corresponding results

This section is devoted to illustrating an example of statistical (Monte-Carlo) calculations where we have used different, randomly generated SPR functions as impedances for upstream and downstream terminations and obtained the dataset of eigen frequencies and growth rates for a system with fixed burner/flame given by its particular TF. The systematic investigation of the obtained dataset and the development of the corresponding analysis procedures are the subjects of future research. Here we will demonstrate a few examples of the possible methods of analysis and research questions that may allow this approach.

The above-described method allows us to randomly choose roots of $q(s)$ and $p(s)$ and obtain a range of SPR functions for each such set of roots. One may also vary the degree of polynomial to add additional randomness to the frequency dependent $RC(s)$ upstream and downstream. This will help in capturing and "probing" an even wider range of different possible boundary conditions that a burner with flame may experience when placed in a real combustion device.

To take into consideration a possibly large phase delay, appropriate to reflecting terminations, the rational function representing $RC(s)$ can be multiplied by a factor of time delay. It allows to mimic the additional effect of upstream and downstream duct lengths incorporated into the reflection coefficients in the following manner:

$$RC_{up}(s) = RC_{up}(s) \cdot \exp\left(-s \cdot {}^{2L_c}/_{c_c}\right)$$
$$RC_{dn}(s) = RC_{dn}(s) \cdot \exp\left(-s \cdot {}^{2L_h}/_{c_h}\right)$$

In the present example the values of additional "transport time delays" are also randomly selected from the range limited by the predefined maximum and minimum values for L_c and L_h as shown in Table 1.

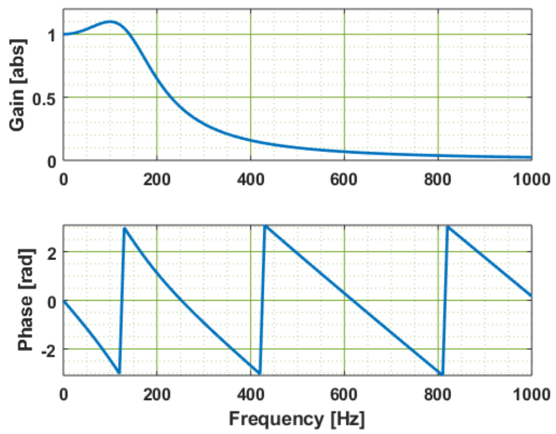
In the example below, we have assumed the following upstream and downstream conditions:

Table 1: Operating Conditions

Section	Length of Duct $L_{section}$	Temperature	Density	Speed of sound
	(m)	(K)	(kg/m^3)	(m/sec)
Cold (c)	0.1-2.0	293	1.204	343.24
Hot (h)	0.1-1.0	1600	0.2207	774.38

To summarize, we are fixing the flame TF in the form of $G(s)$, with which we have calculated our transfer matrix given by $T(s)$. The form of $G(s)$ is given by Equation (5) and shown in Figure 6 (a). We have upstream and downstream reflection coefficients which we obtained from the SPR functions in the form of $RC_{up/dn}(s)$. With all this information we can now calculate the eigen frequency and growth rate by solving Equation (6) for non-trivial solutions.

To obtain the probability of instability and other statistically valuable information about the given flame, we have randomly selected values for roots of $q(s)$, roots of $U(s)$ and $v(s)$, K, upstream duct length (L_c) and downstream duct length (L_h). Obtaining the eigen frequencies for a large number (in this case it is 46,030) of such randomly selected cases, we generated a database which can be used for further analysis.



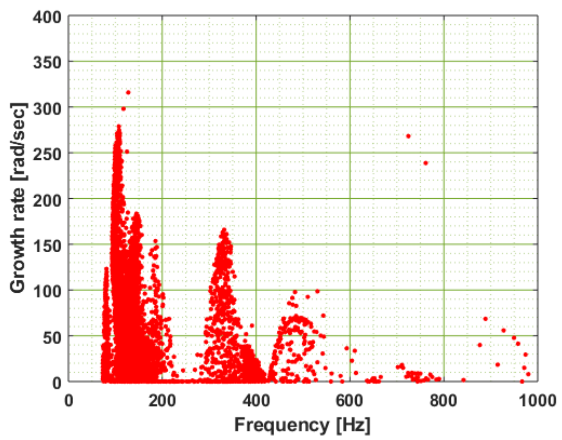


Figure 6: (a) Gain and phase of analytically determined flame Transfer Function and (b) Eigen frequency and positive growth rate obtained.

Figure 6(b) represents an example of possible analysis of data by plotting all eigen frequencies with positive grow rates (12647). From this plot, one can judge about the most probable instability frequencies of the given burner/flame. In this example, the highest concentration of cases with positive growth rate occur around $100 - 150$ Hz. By comparing this plot with the flame TF, one sees that it is around this frequency the phase passes through π, gain of the TF is high and therefore the role of the burner intrinsic instability mode can be significant.

The other possible route of the Monte-Carlo dataset analysis is by plotting the unstable eigen frequencies vs some system parameter. For instance, when we plot the unstable eigen frequencies with respect to upstream duct length (L_c) as shown in Figure 7, the plot reveals that the maximum number of cases where the system is unstable occurs for L_c ranging from $1\,m$ to $2\,m$. Thus, to get another perspective of the effect of L_c on the probability of instability, we have compared all the cases of randomly selected upstream lengths with those cases where the system in unstable (growth rate is positive) as shown in Figure 8. The trend in upstream length shows that if L_c is kept less than $0.7\,m$, then the system will have higher probability of stability than cases when the upstream duct is more than $1\,m$. For example, if the range for length of upstream duct is fixed at $0.46 - 0.51\,m$, then out of 1183 random cases, only 14 cases were unstable giving the probability of instability as $14/1183 = 0.012$. These 14 cases depend on sound reflection at the downstream termination. Whereas if the upstream length is fixed around $1.76 - 1.81\,m$, then from 1177 tested configurations, 998 cases had positive growth rate, therefore the probability of instability becomes 0.85. Clearly, if we were to design an appliance having flame TF similar to the one given by Figure 6(a), we should aim to have the upstream duct length less than $0.5\,m$ to decrease the possibility of system instability.

This kind of analysis with respect to K and downstream length can help us determine which flame TF (experimentally obtained for different burners) is more stable in all configurations of upstream and downstream boundary conditions.

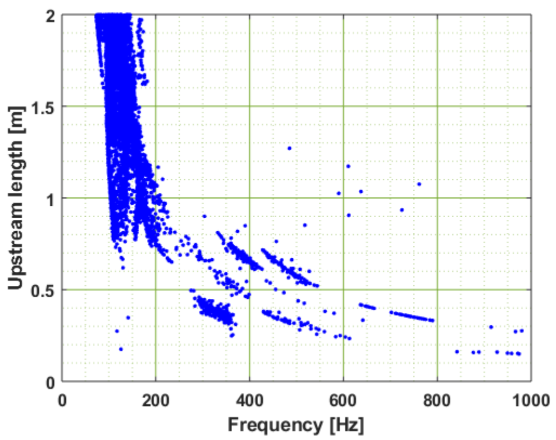


Figure 7: Eigen frequency plotted with respect to upstream length for cases with positive growth rate.

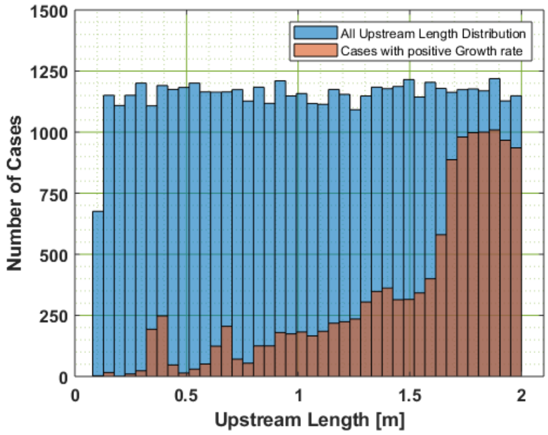


Figure 8: Randomly chosen upstream length compared with results for which the growth rate was positive.

Conclusion

The present contribution is a constituting part of the research program with the ultimate goal to develop a methodology to characterize thermo-acoustic properties of burners with flame in terms of their thermo-acoustic figure of merit. Within the present research line, the idea to perform a Monte-Carlo type of simulation of a generic combustion appliance is followed. The focus of the present contribution is to generate random acoustic embeddings for an active flame that satisfy the requirement of acoustic passivity and statistically represent typical combustion appliances. The acoustic embeddings with impedances described by the strictly positive real functions in terms of complex frequencies are known to fulfil requirements of passivity.

Using the network modeling approach the statistically representative dataset of complex eigen frequencies of a system consisting of a fixed burner/flame and random, frequency dependent acoustic embeddings is produced. Several possible methods of statistical analysis of the generated dataset are proposed and demonstrated. Namely, the probability of instability can be estimated and may serve as the measure of the burner figure of merit. Furthermore, the obtained data can be used to elucidate the effect of specific parameters of burner transfer function or acoustic embedding on the probability of instability.

Based on the conducted study, the following can be concluded.

- The method to generate SPR functions proposed in [6] effectively produces a wide range of acoustic embeddings with properties required for the elucidation of statistically relevant features of acoustic instability of given burner with flame;
- The dataset of eigen frequencies produced by using Monte-Carlo simulation of acoustic network with burner and random reflections may serve for evaluation of many practically relevant parameters of the burner, namely its figure of merit.
- The detailed analysis of the dataset provides new inside into the role of different system parameters on the thermo-acoustic instability of the combustion system.

Next steps in the direction of the present research will include the evaluation of figure of merit for experimentally characterized burners with flames, elucidation of different burner parameters on their thermo-acoustic quality factor. Furthermore, the range of possible analysis method of the (in)-stability statistics will be extended.

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