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EXPERIMENTAL UNCERTAINTY ANALYSIS OF IMPEDANCE MEASUREMENTS

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The study of the interaction between acoustics and flow has a long history, and has attracted many scholars. Consequently the major contributors to acoustic-flow interaction are known and improvements made on models to describe the various interactions, give a relative small impact to the overall effect. To validate these new models, experiments have to be more precise and accurate, otherwise no valid statement can be made if the measurements are agreeing with the improved models.

The flow-acoustic interaction is commonly measured using impedance tubes through which a flow flows. In this paper, a linear uncertainty analysis is presented to determine the precision of the obtained impedance results. Such kind of analysis has been already reported in literature, but in this paper, the analysis has been expanded to include more uncertain variables and the method is investigated to show the limitations of such an analysis. As an application of the analysis, the measurement of a known impedance without flow has been analyzed, revealing the presence of bias errors in the measurement setup.

Introduction

The study of the interaction between acoustics and flow has a long history, and has attracted many scholars. Consequently the major contributors to acoustic-flow interaction are known and improvements made on models to describe the various interactions, give a relative small impact to the overall effect. To validate these new models, experiments have to be more precise and accurate, otherwise no valid statement can be made if the measurements are agreeing with the improved models.

Therefore an essential ingredient in experimental investigations to validate models is to determine the precision and accuracy of the measurement results. Precision gives information on the repeatability of the measurements and accuracy gives information of the correctness of the results. In general the precision of measurements is easier to quantify, in contrast with the accuracy, which can only be determined by measuring known values.

In this study the precision of a measurement is estimated using a multi-variate analysis. In such an analysis, knowledge of the errors on the measurement are taken into account to determine the variance of the measurement results. The variance can then be used to calculate the interval around the measurement results where the true value should lie with a certain probability. Also, the variance can be analysed to determine the contribution of each error and identify which aspects of the measurements have to be improved to obtain more reliable results.

The analysis will be applied to measurements of the impedance of a steel plate. The impedance of the plate is known analytically by taking into account the acoustic boundary layer [1]. With the help of the uncertainty analysis, the precision of the results can be identified and knowledge about the precision of the measurement can be inferred from the comparison of the model results and the confidence intervals.

First the measurement setup and method will be introduced. Then the concepts of the multivariate uncertainty analysis are given, followed by the experimental results. The results are presented together with confidence intervals and the discussion will show which information can be obtained from the uncertainty analysis.

Measurement setup and method

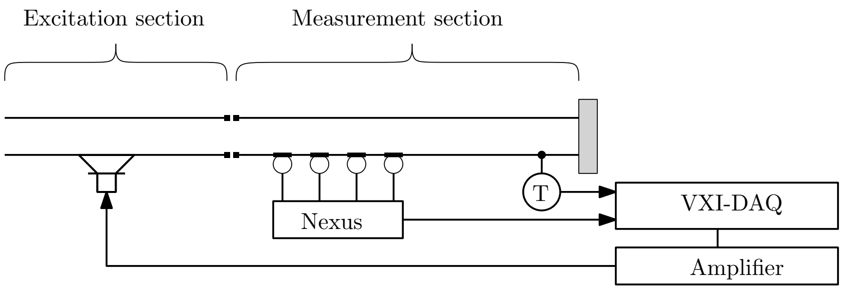

Figure 1: Schematic overview of the experimental setup used to determine the reflection coefficient of a rigid plate.

The measurement setup consists of a wave guide which is terminated by the measurement object, the rigid plate. A schematic overview of the measurement setup is given in Fig. 1. The wave guide is made of aluminium, is cylindrical and has an inner diameter of 50mm and wall thickness of 5mm. The rigid plate is made from steel and has a thickness of 3.5cm. The acoustic wave guide has been disconnected between the loudspeaker section and measurement section to reduce the influence of mechanically vibrations, induced by the loudspeakers. Also, the measurement section itself has been suspended in mid air using chains and rubber bands to ensure that no other mechanical vibrations could affect the measurement. The acoustic excitation was provided by a loudspeaker attached to the excitation section, far from from the measurement section. A stepped sine excitation was used to obtain a high signal-to-noise ratio and the excitation order in terms of frequency was randomized.

The pressure fluctuations were registered by four flush mounted Brüel and Kjær 1/4-inch condenser microphones of type 4938. The microphones were attached to a Brüel and Kjær NEXUS signal conditioner. The positions relative to rigid plate are given in Table 1. The microphones have been calibrated in gain and phase relative to each other by exposing all the microphones to the same sound field in a calibrator [2]. The temperature of the measurement section has been continuously monitored by attaching a thermo-couple to the outside of the waveguide. The acquisition of the measurement signals and the excitation of the loudspeakers were controlled by a HP-VXI system.

The frequency of the acoustic perturbations was lower then the first cut-on frequency of the duct, and thus the sound field in the duct can be represented as two waves travelling in opposite direction [1]. Under these circumstances, the pressure along the duct axis can be written as a function of the wave numbers of the two travelling waves and the amplitude of these waves, i.e.

$$p(x) = p^+ \exp(-ik^+ x) + p^- \exp(+ik^- x), \tag{1}$$

where p is the complex pressure at position x along the duct. The complex wave numbers k^+, k^- are the wave numbers for the waves propagating towards, p^+, and away, p^-, from the system or material under study respectively. The complex wave numbers k^+, k^- are modelled and thus dependent on which physics that are taken into account, such as viscous thermal effects at the walls, losses within the fluid and convective effects. In this study, the experiments are performed in quiescent media, resulting in that $k^+ = k^-$. The wave numbers are determined by including the effect of the wall on the wave propagation up to second order [3] and the influence of ambient condition on the speed of sound are taken into account [4][5].

By measuring the pressure at at least two positions, the two travelling wave components can be deduced by writing the resulting set of equations in matrix form, where the matrix relating the travelling wave components and the measured complex pressures is called the modal matrix,

$$\underbrace{\begin{bmatrix} \exp(-ik^+x_1) & \exp(+ik^-x_1) \\ \vdots & \vdots \\ \exp(-ik^+x_n) & \exp(+ik^-x_n) \end{bmatrix}}_{\text{Modal matrix}} \begin{bmatrix} p^+ \\ p^- \end{bmatrix} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}. \tag{2}$$

and solving the (overdetermined) system of equations (in the linear-least-squares sense) [6, 7]. The reflection coefficient is then given by

$$\mathcal{R} = \frac{p^+}{p^-} \tag{3}$$

To improve the robustness of the measurement results to external noise, the transfer function between the reference signal driving the loudspeaker and the signal from the microphone has been measured, instead of using directly the Fourier transform of the measurement signal [8].

To be able to calculate the uncertainty on the reflection coefficients, the uncertainties on the measured parameters have to be known. In the case of the reflection coefficients, the following parameters have to be measured to be able to calculate the reflection coefficient: the transfer function between the microphone signal and the reference, the positions of the microphones x_n and the parameters that govern the wave number k. These parameters are the ambient humidity RH, the ambient pressure p_{amb}, the ambient temperature T and the pipe diameter D, needed to be able to calculate the damping of the acoustic waves through the pipe. Two principal ways exist to determine the uncertainty, the first is to infer it from technical data and/or experience, the second way is to determine it by measuring a known quantity a multiple times. The measured transfer function is complex and in general the uncertainty has to be represented using a covariance matrix, as will be shown in the next section. In this study, for simplicity it is assumed that the real and imaginary part have the same uncertainty and there is no correlation between the real and imaginary parts. This leads to that the uncertainty on the transfer function can be represented by a single number. For the analysis, a relative error of 1% of the absolute value of the transfer function is assumed, based on the data from the manufacturer.

In Table 1 a list of the measured parameters is given, together with their mean values, uncertainties and how the uncertainty has been determined. For more information about how the uncertainty has been determined, the reader is referred to [9].

Uncertainty analysis

The objective of an uncertainty analysis is to determine the effect of measurement errors on the uncertainty of the parameter of interest, such as the reflection coefficient $\mathcal{R}$. Various ways exists to

Table 1: Uncertainty table of the measurement

Variable	Mean Value	Standard uncertainty	Source
Amb. humidity	25% [RH]	3.23% [RH]	Manufacturer
Amb. pressure	100 [kPa]	1.30 [hPa]	Manufacturer
Amb. temperature	25 [°C]	0.0675 [°C]	Measurement
Pipe diameter	50 [mm]	65.2 [μm]	Manufacturer
Transfer function	-	1% rel. error	Manufacturer
Microphone Position 1	0.4816 [m]	8.62e-2 [mm]	Measurement
Microphone Position 2	0.5498 [m]	0.98e-1 [mm]	Measurement
Microphone Position 3	0.5847 [m]	1.05e-1 [mm]	Measurement
Microphone Position 4	0.6113 [m]	1.44e-1 [mm]	Measurement

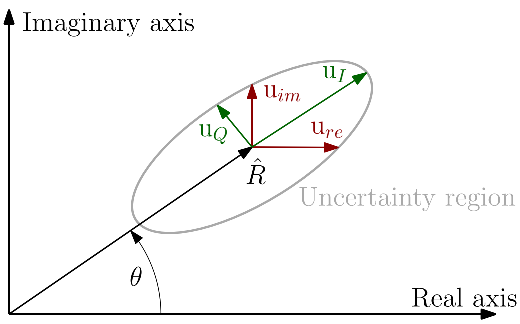

Figure 2: Schematic overview of the uncertainty region of a measured phasor in the complex domain, described by the covariance matrix. Two representations are shown, the uncertainties described by the real and imaginary parts in green and the uncertainties in in-phase and quadrature components in red

determine the uncertainty on a measurement and the two most used methods are the so-called Monte Carlo method [10], where the uncertainty is numerically determined and the multi-variate analysis [10], where the uncertainty is determined using a Taylor expansion. The latter method will be used in this study as the problem under consideration is in approximation linear.

In general, the measurements $\boldsymbol{x} = [x_1, x_2, ...x_n]^T$ are related to the parameter of interest q by a function f,

$$q = f(\boldsymbol{x}). \tag{4}$$

The parameter of interest, q, can be a vector, but for simplicity in notation it is considered a scalar and in our case, the wanted parameter is the reflection coefficient $\mathcal{R}$ and the measured parameters, measurands, are given in Table 1. The function f is given by the Eqs. (2) and (3). For this study, some of the measurands and the parameter of interest are complex. Therefore, the uncertainty on q consist of both a real and complex part and also the correlation between these part. This information can be represented using a covariance matrix, $\boldsymbol{v}(q)$ [11, 12].

The covariance matrix of the variance of the parameter of interest is given by,

$$\boldsymbol{v}(q) = \sum_{i=1}^{N}\sum_{j=1}^{N} \boldsymbol{u}_i(q)\boldsymbol{r}_{ij}\boldsymbol{u}_j(q), \tag{5}$$

wherein is $\boldsymbol{u}_i(q)$ the component of uncertainty matrix and $\boldsymbol{r}_{ij}$ the correlation matrix. The component of uncertainty matrix gives the contribution of each uncertain measurand to $\boldsymbol{v}(q)$ and is given by,

$$\boldsymbol{u}_i(q) = \boldsymbol{J}_i\boldsymbol{u}(x_i). \tag{6}$$

Herein is $\boldsymbol{J}_i$ the Jacobian matrix of the wanted parameter q with respect to the i-th complex input parameter and $\boldsymbol{u}(x_i)$ a diagonal matrix with the uncertainties of the real and imaginary parts on the diagonal.

The correlation matrix $\boldsymbol{r}_{ij}$ gives the correlation between the real and imaginary parts of the i-th and j-th measurand. If $i = j$, the correlation matrix is symmetric and the diagonal is equal to $[11]$ and the off-diagonal components are equal to the correlation coefficient between the real and imaginary part of the measurand.

To calculate the uncertainty intervals for the determined impedance, some special considerations have to be taken into account. Due to the determination of the complex impedance, a correlation between the real and imaginary part of the impedance is created. This means that in the complex domain, the variance of the impedance is defined by a elliptic region in contrast with that of an interval for real valued problems. Furthermore, as often the complex impedance is described using the reflection coefficient, which is a phasor, the real and imaginary uncertainty vectors have to be transformed to the in-phase and quadrature uncertainty components of the measured phasor. A schematic representation is given in Fig. 2. This transformation can be taken into account by rotating the frame of reference [13] w.r.t to the angle θ of the measured phasor,

$$\boldsymbol{\Sigma}_{IQ} = \boldsymbol{\Theta}(-\theta)\boldsymbol{\Sigma}_{RI}\boldsymbol{\Theta}(\theta), \tag{7}$$

wherein $\boldsymbol{\Sigma}_{IQ}$ is to covariance matrix in the quadrature and in-phase uncertainty components, $\boldsymbol{\Sigma}_{RI}$, covariance matrix in real and imaginary uncertainty components and $\Theta(\theta)$ the rotation matrix given by,

$$\boldsymbol{\Theta}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{8}$$

As a special case, when the amplitude of the measured reflection coefficient $\mathcal{R}$ is large compared to the in phase and quadrature uncertainty component, then the variance can be expressed in terms of phase σ_θ^2, magnitude σ_m^2 and the correlation between these variances $\rho_{\theta m}$,

$$\boldsymbol{\Sigma}_{IQ} = \begin{bmatrix} \sigma_m^2 & \sigma_m\sigma_\theta\rho_{\sigma m}|\hat{\mathcal{R}}| \\ \sigma_m\sigma_\theta\rho_{\sigma m}|\hat{\mathcal{R}}| & |\hat{\mathcal{R}}|^2\sigma_\theta^2 \end{bmatrix}. \tag{9}$$

If the covariance matrix on the measured parameter is known, the confidence intervals can be calculated if the underlying distribution for that parameter is known. In this study, it is assumed that the underlying distribution of the output parameter is normal distributed to show the concepts. However in general, this statement should not be made a-priori and the distribution should be estimated [11][12].

For a normal distribution, the interval where the true value of the parameter x lies is given by the measured value $\bar{x}$, the standard deviation of the parameter σ and the probability p that the true value will lie within this interval,

$$\bar{x} - n(p)\sigma < x < \bar{x} + n(p)\sigma \tag{10}$$

herein $n(p)$ is the coverage factor and dependent on the probability p that the true value will be in the given interval. For a 65 % probability $n = 0.93$, and the 95 % confidence interval is given by $n = 1.96$.

On the linear propagation of errors

The multi-variate analysis is only appropriate when the error propagation from the measurement to the parameter of interest is linear. It is argued that one should prefer to use the Monte-Carlo method to determine the uncertainty of an impedance measurement with the two-microphone method [14], because the error propagation can be non-linear under some circumstances. As the multi-variate analysis gives more insight and is computationally faster, the question arises under which circumstances this linear propagation is justified.

Non-linear relations between the error on the measurands and the parameter of interest are formed in two ways when determining the travelling wave components in Eq. (2). The first is that linear errors on the modal matrix coefficients can lead to non-linear errors on $[p^+, p^-]^T$ by the inverse operation and the second path way is that linear errors in the wavenumber k and position x can lead to non-linear disturbances in the coefficients of the modal matrix because the coefficients have an exponential dependence on these parameters.

For the first path, consider the inverse of a square matrix A with a small linear perturbations $\epsilon\boldsymbol{X}$, The Taylor expansion on ϵ of the inverse is given by,

$$(\boldsymbol{A} + \epsilon\boldsymbol{X})^{-1} = \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{X}\boldsymbol{A}^{-1}\epsilon + \boldsymbol{A}^{-1}\boldsymbol{X}\boldsymbol{A}^{-1}\boldsymbol{X}\boldsymbol{A}^{-1}\epsilon^2 + \mathcal{O}(\epsilon^3) \tag{11}$$

From this equation, a conservative condition for linear error propagation can be found by comparing the size of the norm of the second and third term in the Taylor expansion,

$$\beta_{\boldsymbol{A}} = \frac{||\boldsymbol{A}^{-1}\boldsymbol{X}\boldsymbol{A}^{-1}\boldsymbol{X}\boldsymbol{A}^{-1}\epsilon^2||}{||\boldsymbol{A}^{-1}\boldsymbol{X}\boldsymbol{A}^{-1}\epsilon||} \leq ||\boldsymbol{X}||\,||\boldsymbol{A}^{-1}||\,|\epsilon| \tag{12}$$

By considering all the perturbations where the perturbation $\epsilon\boldsymbol{X}$ leads to a relative change of ϵ of the size of the matrix $\boldsymbol{A}$, measured by the norm, it follows that $||\boldsymbol{X}|| = ||\boldsymbol{A}||$. Under these considerations the ratio between the second and third term in the Taylor expansion reduces to:

$$\beta_{\boldsymbol{A}} \leq ||\boldsymbol{A}||\,||\boldsymbol{A}^{-1}||\,|\epsilon| = k(\boldsymbol{A})|\epsilon|, \tag{13}$$

where $k(\boldsymbol{A})$ is the condition number of the matrix. It shows that if $k(\boldsymbol{A})$ is large, even small errors in the modal matrix may lead to non-linear errors in the solution of Eq. (2).

For the second path, the coefficients of the matrix A itself are linear with respect to the input. As can be seen in Eq. (2), the coefficients all have the similar form of $e^{\pm ikx}$. The ratio between the first and second Taylor term of the expansion in the error of the wavenumber $k + \epsilon_k$ and in the error of the position $x + \epsilon_x$ are given by $\beta_x = i\epsilon_x k$ and $\beta_k = i\epsilon_k x$ respectively. These ratios should also be smaller than one, such that the errors in the coefficients are linear related to the input errors. The size of β_x and β_k is linearly proportional to k and x respectively and it shows that even if one has very small measurement error, these errors are magnified by the factors k and$x$, resulting in a non-linear relation between the measurement error and the error on the travelling wave components.

In summary, if $\beta_x \ll 1$, $\beta_k \ll 1$ and $\beta_{\boldsymbol{A}} \ll 1$ then linear propagation of the error is justified and the multi-variate analysis is appropriate to determine the errors on the travelling wave components. If these factors become large, a linear assumption is not sufficient and second order terms in the uncertainty Eq. (5) must be taken into account. Including the third order and higher terms becomes non-trivial and it is best to resort to numerical methods, such as the Monte Carlo method, to determine the uncertainty [10].

Results

In this section the results from the measurement of the reflection coefficient of the rigid plate will be shown together with the obtained confidence intervals using the multi-variate analysis. Furthermore, the contribution of each error in the measurands to the total variance will be shown and discussed. In Fig. 3, the measured reflection coefficient is shown as function of frequency. The reflection coefficient, with the inclusion of sound absorption due to the viscous thermal acoustic boundary layer should be very close to unity. At 4 kHz, the reflection coefficient is predicted to be

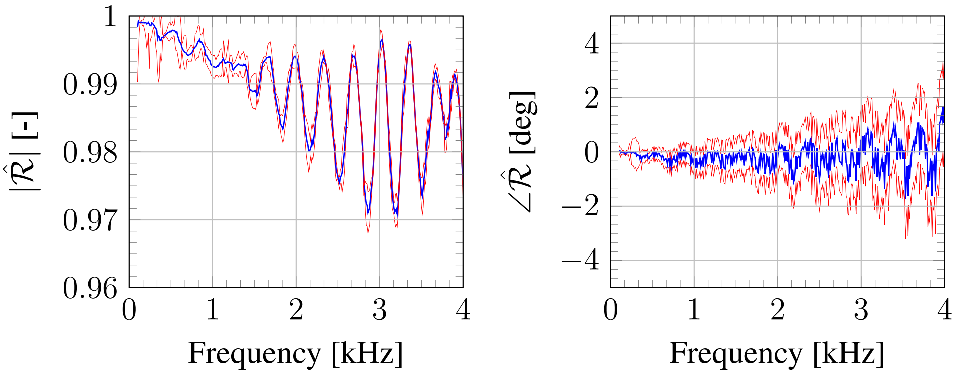

Figure 3: Measured reflection coefficient of the rigid plate. In blue the measured value and red gives the 95% confidence intervals

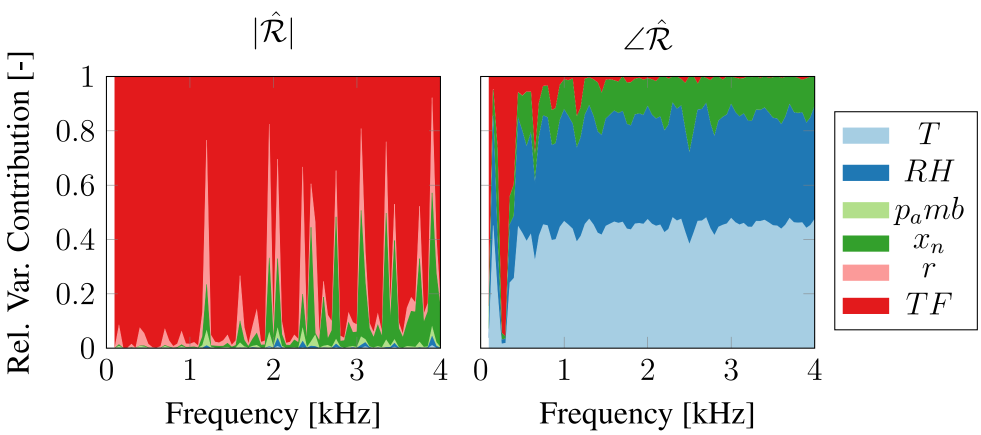

Figure 4: Relative contribution of various errors sources to overall variance. The uncertainty sources are errors related to the temperature, T, the relative humidity, RH, the ambient pressure, p_amb, the microphone positions, x_n, the pipe radius r and the measured transfer function TF

$\mathcal{R} = 0.9986\angle - 0.07\text{deg}$ [1]. From the measurements it can be directly seen that there is a deviation with respect to the model results.

In the figure, the uncertainty intervals for the measurement results are also shown. These uncertainty intervals give the range in which the true parameter will ly, based on the uncertainty analysis. It can be seen, that for the phase the model results lies within the uncertainty interval, showing that one cannot distinguish if the error is due to a known error based on the uncertainty analysis, or if the error is an error which has not been accounted for within the modelling.

On the other hand, for the reflection coefficient, there is an oscillation present in the results, and the confidence intervals closely follow this oscillation. This results show that there is a significant deviation from the true value, based on models, and that this cannot be caused by any of the identified errors within the uncertainty analysis. The models for the wave propagation have been validated to great precision, [15] and therefore the results show that there is an effect that has not been modelled in Eq. (1) and Eq. (2). It is interesting to note that other authors have observed similar discrepancies [16].

The uncertainty analysis itself identifies the parameters that have to be improved to obtain better precision in the reflection coefficient. In Fig. 4 the contribution of each error source to the total variance is plotted as an area plot as function of frequency, in these plots the total variance has been normalized. From this plot, the relative contribution of each error source can be immediately identified. As an example, the precision of the measured phase of the reflection coefficient is governed by the errors on the temperature and relative humidity. Therefore to improve the measurement results regarding the phase of the reflection coefficient, these parameters have to be measured more accurately. On the other hand, if the goal is improve the absolute value of the reflection coefficient, the error is dominated by the error on the measured transfer functions and to improve the measurement results, the error on the measured transfer functions has to be reduced.

Conclusion

In this study, the concepts and use of a multi-variate analysis have been demonstrated on the determination of the acoustic impedance of a steel plate.

The multi-variate analysis has been introduced and the extension to the complex domain have been discussed. The method is based on a linear approximation of the measurement results on the wanted parameter and guidelines to the validity of the linear approximation have been derived. It has been shown that three conditions have to be fullfilled to have a linear error propagation and that errors in the coefficients of the modal matrix can behave non-linearly even tough that the measurement error itself is small compared to the mean value.

The uncertainty analysis has been applied to measurements where the reflection coefficient of a rigid plate was determined. Using the results of the analysis, it has been shown that there is discrepancy between the measurement results and theoretical results, which can not be explained by the current knowledge of the measurement error. Furthermore, the analysis has been used to determine the relative contribution of each error source to the overall error, indicating which error sources have to be reduced to improve the measurement precision.

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