



The acoustic equivalence of a mass and heat source

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In combustion systems, unsteady heat release acts as a source term to the acoustic field within the combustor and under the right conditions the energy of the acoustic field can exponentially grow, leading to a combustion instability. An acoustic driver such as a loudspeaker or horn also acts as a source term to the acoustic field and is often modelled as a fluctuating mass source. Considering the similarity of the flame and the acoustic driver to acts as a source to the acoustic field, the question arises if these two types of sources can be interchanged.

This contribution investigates that question by considering a 1-D system with mean flow. In the governing equations a mass source term is included which is linearly related to velocity fluctuations. The results are compared with that of a system with a compact heat source. It is found that the two systems are equivalent when there is no mean flow. In the presence of flow, the response of both systems can approximately be the same when special conditions are met.

I. Introduction

COMBUSTION instabilities are a concern in the design and commissioning of combustion equipment. These instabilities are governed by a complex interaction between acoustic fluctuations, unsteady heat release and hydrodynamics in the combustor and may cause undesirable noise, vibrations and local thermal and mechanical stresses. In the last decades, several methods have been explored to predict the occurrence of these thermo-acoustic instabilities which can be used to avoid thermo-acoustic instabilities in the early design phase.^{1,2}

The instabilities arise when the so called Rayleigh criterion is met.³ When the criterion is met, the unsteady heat release of the flame is feeding energy into the acoustic field, leading to an exponential growth of the acoustic energy. For laminar premixed flames, the unsteady heat release is primarily a function of velocity fluctuations and this relation can be described using the so called flame transfer function.^{4,5} Similarly, an acoustic driver such as a horn or loudspeaker can also feed energy into the acoustic field. If these two types of acoustic sources can be equivalent under some circumstances, it may be possible to investigate the stability of the combustor by mimicking the acoustic source term of the flame by an acoustic driver.

This contribution investigates the equivalence of the two sources by studying a 1-D system with a mean flow and compact sources. First the governing equations are derived with the inclusion of a mass source term, representing the effect of an acoustic driver, and a heat source, representing the effect of a flame. Then the relation between the acoustic field up and downstream are derived for both the mass source and heat source respectively. The derivation is done using a methodology based on the Rankine-Hugoniot jump conditions. Thereafter the relations are linearised and the source terms are linearly related to the upstream perturbations. As the last step, the linear relations are condensed to a matrix form and the expressions compared.

II. Governing equations

IN this section, the equations for the mass conservation, momentum conservation and energy conservation equation are derived with the inclusion of a mass source term. The purpose of the section is to show which terms

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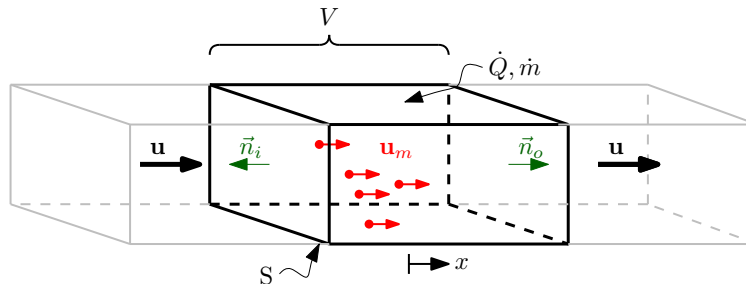


Figure 1. Schematic representation of the control volume with a mass source and a heat source

have to be added to be able to correctly represent the influence of a mass source. First, the integral equations for the mass conservation, momentum conservation and energy conservation are derived with the inclusion of a mass source term. In figure 1 a schematic of the problem is sketched. We consider a control volume V , located in a region with a uniform mean flow with velocity \mathbf{u} . In the control volume two sources are present, a heat source of strength \dot{Q} and a mass source of strength \dot{m} . Furthermore, the added mass has a velocity of \mathbf{u}_m . The normal, pointing away from the surface of the control volume, is indicated with \mathbf{n} . For the mass conservation equation there will be an extra source term, given in red, due to the added mass,

$$\frac{\partial}{\partial t} \iiint_V \rho dV + \iint_S \rho \mathbf{u} \cdot d\mathbf{S} = \iiint_V \dot{m} dV. \quad (1)$$

Herein t is the time, ρ is the density, \dot{m} is the specific mass addition per unit time ($\text{kg}/\text{m}^3\text{s}$), $d\mathbf{S} = \mathbf{n}dS$ is the infinitesimal area dS multiplied with the normal vector, \mathbf{n} , directed outwards from the volume and \mathbf{u} is the velocity.

The momentum conservation equation is based on that the time rate of change of momentum of a body equals the net force exerted on it. If the added mass has a velocity component, the addition will also act as a source in the momentum equation in the time rate change of momentum. The momentum equation, with the inclusion of a momentum source term becomes,

$$\iint_S (\rho \mathbf{u} \cdot d\mathbf{S}) \mathbf{u} + \iiint_V \frac{\partial (\rho \mathbf{u})}{\partial t} dV - \iiint_V \dot{m} \mathbf{u}_m dV = \iiint_V \rho \mathbf{f} dV - \iint_S p \cdot d\mathbf{S}. \quad (2)$$

Herein \mathbf{f} is the body force acting on the fluid within the control volume, p is the pressure acting on the surfaces of the control volume and \mathbf{u}_m is the velocity associated with the added mass. If the added mass does not have a velocity component associated with it, the added mass will lead to a decrease of the specific momentum within the control volume and thus the mass source acts as a sink term in the momentum equation.

The last conservation equation that is needed to describe the system is the energy equation. In the control volume, energy in the form of heat is added by the source \dot{Q} . Furthermore, the energy in the control volume increases because the added mass has internal and kinetic energy associated to it. This leads to that the energy equation is given by,

$$\begin{aligned} \iiint_V \dot{Q} dV - \iint_S p \mathbf{u} \cdot d\mathbf{S} + \iiint_V \rho (\mathbf{f} \cdot \mathbf{u}) dV + \iiint_V \dot{m} \left(e_m + \frac{1}{2} \mathbf{u}_m^2 \right) dV = \\ \iiint_V \frac{\partial}{\partial t} \left[\rho \left(e + \frac{u^2}{2} \right) \right] dV + \iint_S \rho \left(e + \frac{u^2}{2} \right) \mathbf{u} \cdot d\mathbf{S}, \end{aligned} \quad (3)$$

where \dot{Q} is the specific energy added per unit time [$\text{J}/\text{kg s}$]. The internal energy of the fluid is represented by e and the internal energy of the added mass is given by e_m .

Using Gauss' theorem, we can rewrite the above obtained integral conservation equations in differential form,

$$\nabla \cdot \rho \mathbf{u} + \frac{\partial \rho}{\partial t} = \dot{m}, \quad (4)$$

$$\nabla \cdot (\rho \mathbf{u} \mathbf{u}) + \frac{\partial (\rho \mathbf{u})}{\partial t} - \dot{m} \mathbf{u}_m = \rho \mathbf{f} - \nabla p, \quad (5)$$

$$\dot{Q} - \nabla \cdot (p \mathbf{u}) + \rho (\mathbf{f} \cdot \mathbf{u}) + \dot{m} \left(e_m + \frac{1}{2} \mathbf{u}_m^2 \right) = \frac{\partial}{\partial t} \left[\rho \left(e + \frac{u^2}{2} \right) \right] + \nabla \cdot \rho \left(e + \frac{u^2}{2} \right) \mathbf{u}. \quad (6)$$

The final equation that has to be used to close the system of equations is the equation of state. In the analysis, the fluid in the control volume is considered to be an ideal gas and the relation between the pressure p , density ρ and temperature T is given by the ideal gas law,

$$p = \rho RT, \quad (7)$$

where R is the ideal gas constant. Further more, the fluid is considered to be thermally and calorically perfect,⁶ meaning that the specific heat at constant pressure, c_p , and volume, c_v , are constant and that the internal energy is linearly dependent on the temperature of the gas,

$$e = c_v T. \quad (8)$$

The ideal gas constant R is related to the specific heats through $R = c_p - c_v$ and the ratio between the specific heats is given by $\gamma = c_p/c_v$.

In this study, the two variables describing the state of the gas are chosen to be the pressure p and the internal energy e . When a heat source is present it is often customary to use the entropy, s , as state variable instead of the internal energy. However, with the presence of the mass source, the energy equation written in terms of entropy is given by:

$$\rho T \frac{Ds}{Dt} = \dot{Q} + \dot{m} \left(e_m + \frac{1}{2} |\mathbf{u}_m|^2 \right) - \dot{m} \left[e - \frac{1}{2} |\mathbf{u}|^2 + \mathbf{u} \cdot \mathbf{u}_m + \frac{p}{\rho} \right]. \quad (9)$$

Note that in this case, the entropy is not longer a function of only the source terms \dot{m} and \dot{Q} , but also the other variables.

A. Rankine-Hugoniot jump conditions

With the conservation equations including the mass source terms, the linking conditions between the upstream and downstream side of the mass source can be derived. Consider a 1-D domain with a spatial length of $x \in$

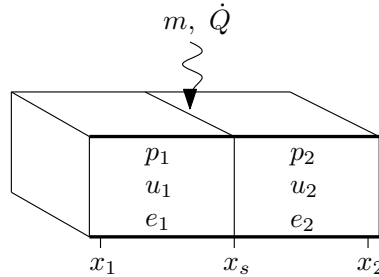


Figure 2. Schematic of the jump conditions

$[x_1, x_2]$. The domain is subdivided in two regions, $x \in [x_1, x_s]$ and $x \in [x_s, x_2]$, where the governing quantities, pressure, p , velocity, u , and internal energy, e , are conserved, but there is also a source present at x_s , as depicted in figure 2. Assuming that the extent of the source in the principal direction is much smaller than the length scale (wave length) of the problem, the change in state variables from state 1 to 2 can be seen as a discontinuity at x_s where the source term is present. Using the above reasoning, the mass and heat sources will be represented as point sources located at the discontinuity $[\dot{Q}, \dot{m}] = [\dot{Q}, \dot{m}] \delta(x_s - x)$, with δ the delta function. Furthermore there are no external body forces acting on the domain ($\mathbf{f} = 0$).

The governing variables left and right of the source show a discontinuity and the relations between the governing variables upstream and downstream of the source are called the jump conditions. To be able to derive

the jump conditions, the conservation equations should be in the form of $\partial\psi/\partial t + \nabla \cdot \psi = \delta s$, where ψ is the conserved quantity and s the source term. If one would choose the entropy representation, eq. (9), the equations are not in the right form, leading to difficulties to derive the jump conditions. The jump conditions are obtained by integrating the differential equations (6) over the volume between x_1 and x_2 and splitting the integrals at the source boundary. Assuming that the source boundary is non-stationary, $x_s = f(t)$, the integrals can be rewritten using the Leibniz integral rule. Then invoking that the source position is stationary, $\frac{dx_s}{dt} = 0$, the following relations are obtained: the mass conservation equation,

$$\rho_2 u_2 - \rho_1 u_1 = \dot{m}, \quad (10)$$

the momentum equation,

$$\rho_2 u_2^2 - \rho_1 u_1^2 + p_2 - p_1 = \dot{m} u_m, \quad (11)$$

and the energy equation,

$$p_2 u_2 - p_1 u_1 + \rho_2 e_2 u_2 - \rho_1 e_1 u_1 + \frac{1}{2} \rho_2 u_2^3 - \frac{1}{2} \rho_1 u_1^3 = \dot{m} (e_m + u_m^2/2) + \frac{\dot{Q}}{A}. \quad (12)$$

Herein \dot{Q} is the rate of heat added to the system [J/ s], A is the cross-section area of the duct and \dot{m} is the rate of mass added to the system in [kg/s]. The subscript 1 denotes the state upstream of the source and the subscript 2 denotes the state downstream of the source. The time dependence has vanished, because of the stationary position of the source location and the infinite smallness of the domain $[x_1, x_2]$. This means that the results derived from the analysis will be a quasi steady state solution or equivalently the angular frequency of the perturbations $\lim \omega \rightarrow 0$.

The next step is to linearise the equations by assuming that each variable is composed of a constant part $\bar{\phi}$ and a small perturbation ϕ' , substituting each variable by $\phi = \bar{\phi} + \phi'$ and neglecting terms of second order or higher in the primed variables. Then the equations can be grouped in two parts, the unperturbed and perturbed equations. Together with the ideal gas law for the unperturbed state $\bar{p} = \bar{\rho} \bar{R} \bar{T}$ the equations for the unperturbed state can be rewritten as,

$$\bar{\rho}_2 \bar{u}_2 - \bar{\rho}_1 \bar{u}_1 = \bar{m}, \quad (13)$$

$$\bar{p}_2 (1 + \gamma \bar{M}_2^2) - \bar{p}_1 (1 + \gamma \bar{M}_1^2) = \bar{m} \bar{u}_m, \quad (14)$$

$$\bar{p}_2 \bar{u}_2 \left(1 + \gamma \bar{M}_2^2 + \frac{c_v}{R}\right) - \bar{p}_1 \bar{u}_1 \left(1 + \gamma \bar{M}_1^2 + \frac{c_v}{R}\right) = \bar{m} (\bar{e}_m + \bar{u}_m^2/2) + \frac{\bar{Q}}{A}, \quad (15)$$

where $\bar{M} = \bar{u}/c$ is the Mach number, where c is the speed of sound given by $c = \gamma \bar{R} \bar{T}$. For a zero mean mass influx, $\bar{m} = 0$, and no heat source, $\bar{Q} = 0$, these equations are the Rankine-Hugoniot conditions for a stationary shock.⁶

The perturbed equations are linear in the perturbations and can thus be written as a system of linear equations. By using the linearised equation of state $p' = \bar{\rho} R T' + R \bar{T} \rho'$, the system can be written in the form of,

$$A_2 x'_2 = A_1 x'_1 + M m' + Q q', \quad (16)$$

for which the matrices A_1 and A_2 contain only terms dependent on, respectively, the unperturbed upstream and downstream states. The vector of the perturbations is given by $x' = [p', u', e']^T$ and the subscript on x' indicates to which perturbations are referred to, upstream or downstream of the source. The matrix A_1 is given by,

$$A_1 = \begin{bmatrix} \frac{\bar{u}_1}{\bar{T}_1 R} & \bar{\rho}_1 & -\frac{\bar{\rho}_1 \bar{u}_1}{\bar{T}_1 c_v} \\ 1 + \gamma \bar{M}_1^2 & 2 \bar{\rho}_1 \bar{u}_1 & -\frac{\bar{\rho}_1 \bar{u}_1^2}{\bar{T}_1 c_v} \\ \bar{u}_1 \left(1 + \frac{1}{2} \gamma \bar{M}_1^2 + \frac{1}{\gamma-1}\right) & \bar{p}_1 \left(1 + \frac{3}{2} \gamma \bar{M}_1^2 + \frac{1}{\gamma-1}\right) & -\bar{\rho}_1 \bar{u}_1 \bar{M}_1^2 \gamma (\gamma-1) \end{bmatrix}, \quad (17)$$

and A_2 is obtained by replacing the subscript 1 by 2. The source terms to equation (16) are given by:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ \bar{u}_m & 2 \bar{m} & 0 \\ \bar{e}_m & \bar{u}_m \bar{e}_m & \bar{m} + \frac{\bar{u}_m^2}{2} \end{bmatrix}, m' = \begin{bmatrix} \dot{m}' \\ u'_m \\ e'_m \end{bmatrix}, \text{ and } Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, q' = \begin{bmatrix} \dot{Q}' \\ 0 \\ 0 \end{bmatrix}. \quad (18)$$

III. Reduction of the system

IN this section, the equations governing the relation between the upstream and downstream mean quantities, I eqs. (13)-(15), and perturbed quantities, equations eq. (16), are simplified and rewritten. The relation between the perturbed quantities up and downstream of the source will be represented by a single matrix, for which the coefficients are only dependent on the upstream mean quantities.

Two sets of equations will be derived, one for the mass source and the other for the heat source. First the relationship between the upstream and downstream perturbations without the influence of a fluctuating source is derived and thereafter the sources are linearly coupled to the upstream fluctuations.

The equations have been reduced using an open-source computer algebra system, Maxima.⁷

A. Mass source

To derive the relationships between the perturbations upstream and downstream of the mass source, additional simplifications are made. First, the mass source is assumed to have no constant mass addition, $\bar{m} = 0$, and the added mass does not have any mean velocity associated with it $\bar{u}_m = 0$. Also it is assumed that the Mach number of the flow is small, $\bar{M}_1, \bar{M}_2 \ll 1$ and therefore the higher order terms in Mach-number in (14) and (15) are neglected to determine the relations between the unperturbed states up- and downstream of the source.

The first step is to look at the relation between the upstream and downstream fluctuations without the source terms. With the above assumptions it follows that the unperturbed variables do not have a jump at the source plane, eq. (13)-eq. (15), and thus $A_2^{-1}A_1 = I$.

The second step is to look how the mass source is coupled to the fluctuations of the governing quantities. The idea for this study is to be able to study the stability of a combustor using a mass source instead of a heat source. For a heat source, the source term is linearly coupled with the upstream velocity perturbations.⁵ In the case of the mass source, an actual realization would involve a form of active control, where the upstream fluctuations are measured and leading to an addition of mass at the source plane. In such a system, it would be fairly easy to be able to measure pressure and velocity fluctuations of reasonable frequencies using for example microphones and hot wire anemometry, but temperature fluctuations would be harder to measure due to the thermal inertia of conventional measurement techniques (thermocouple, RTD etc.). With the above reasoning, the mass source is assumed to be only dependent on the upstream pressure and velocity perturbations,

$$m' = m_p p'_1 + m_u u'_1, \quad (19)$$

where m_p and m_u are the proportionality constants between the fluctuating mass source and the upstream fluctuations. Furthermore, assuming that the added mass is in the form of an ideal gas and has the same temperature as that of the fluid in the duct, the added internal energy can be rewritten as $\bar{e}_m = c_1^2/\gamma(\gamma - 1)$.

To determine the relationships between the perturbed states, terms of second order in Mach number have been neglected in (17) for both the upstream and downstream side if their contribution is small compared to the other terms within the specific coefficient. By solving the linear system with the suggested coupling between the mass source and the upstream perturbations, and only retaining first order terms w.r.t to the mean flow velocity in the enumerators and denominators, the relation between the upstream and downstream perturbations is given by:

$$\mathbf{x}'_2 = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -\frac{\gamma+1}{\gamma} \bar{u}_1 m_p & -\frac{\gamma+1}{\gamma} \bar{u}_1 m_u & 0 \\ \frac{c_1^2}{\gamma^2 \bar{p}_1} m_p & \frac{c_1^2}{\gamma^2 \bar{p}_1} m_u & 0 \\ -\frac{c_1^2}{\gamma^3 \bar{p}_1 \bar{u}_1} m_p & -\frac{c_1^2}{\gamma^3 \bar{p}_1 \bar{u}_1} m_u & 0 \end{bmatrix} \right) \mathbf{x}'_1. \quad (20)$$

In the limit of $\bar{u}_1 \rightarrow 0$, the source terms in the energy equation are not defined. This problem is originating from the fact that when there is no mean flow present the last column in A is empty, therefore the matrix does not longer have full rank and thus the inverse is not defined. From a physical point of view, the added energy is not convected or diffused away from the source plane, because there is no mean flow and thermal conduction has been neglected respectively. This results in an energy singularity at the source plane.⁸ This problem can be circumvented by not taking into account the energy equation in the analysis. This can be justified because the mass source is not a function of the energy perturbations and thus if only acoustic disturbances are considered, the fluctuating energy and pressure at each side is related to each other because of the isentropic nature of the acoustic

disturbances. Under these considerations, the system is reduced to only the mass and momentum equation,

$$\mathbf{x}'_2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{c_1^2}{\gamma^2 \bar{p}_1} m_p & \frac{c_1^2}{\gamma^2 \bar{p}_1} m_u \end{bmatrix} \right) \mathbf{x}'_1. \quad (21)$$

B. Heat Source

In the case of a heat source, the mean energy equation (15) has a source term, which introduces an increase of the energy in the downstream section due to an increase of the temperature. Assuming that the Mach number is small, $M_1^2, M_2^2 \ll 1$, the dependency of the Mach number in the unperturbed equations (14) and (15) is not retained and the following relations hold for the unperturbed variables across the flame,

$$\bar{\rho}_2 \bar{u}_2 = \bar{\rho}_1 \bar{u}_1, \quad \bar{p}_1 = \bar{p}_2, \quad \bar{u}_2 = \bar{u}_1 \left(1 + \frac{\bar{Q}}{A \bar{p}_1 \bar{u}_1} \frac{\gamma - 1}{\gamma} \right), \quad (22)$$

and equivalently

$$\frac{\bar{u}_2}{\bar{u}_1} = \frac{c_2^2}{c_1^2} = \frac{\bar{T}_2}{\bar{T}_1} = \theta = 1 + \frac{\bar{Q}}{A \bar{p}_1 \bar{u}_1} \frac{\gamma - 1}{\gamma}, \quad \bar{M}_2 = \theta \sqrt{\theta} \bar{M}_1. \quad (23)$$

The first step is to determine the relation between the fluctuations up and down stream of the source without the influence of a fluctuating source. Using the system of equations describing the fluctuations, eq. 16, and neglecting the source terms, this relation is described by $\mathbf{A}_2^{-1} \mathbf{A}_1$ (eq. 16). This relation can be simplified by assuming that $M_1^2, M_2^2 \ll 1$ and neglecting these terms in the coefficients of \mathbf{A} , eq (17), only if they are small compared to the other terms in the specific coefficient. This leads to,

$$\mathbf{A}_2^{-1} \mathbf{A}_1 = \begin{bmatrix} 1 & -\frac{\gamma \bar{p}_1 \bar{u}_1 (\theta - 1)}{c_1^2} & 0 \\ -\frac{\bar{u}_1 (\theta - 1)}{\bar{p}_1} & 1 & 0 \\ \frac{c_1^2 (\theta - 1)}{\gamma (\gamma - 1) \bar{p}_1} & -\frac{(\theta - 1) c_1^2}{\gamma (\gamma - 1) \bar{u}_1} & \theta \end{bmatrix}. \quad (24)$$

The second step is to introduce the coupling between the source term and the perturbations at the upstream side. For a flame, the fluctuations of heat release rate \dot{Q} are governed by the flame transfer function, FTF,⁵

$$\text{FTF} = \frac{\dot{Q}' \bar{u}_1}{u'_1 \bar{Q}}, \quad \dot{Q}' = \text{FTF} \frac{\bar{Q}}{\bar{u}_1} u'_1. \quad (25)$$

Combining this with the expression for the mean heat release \bar{Q} , (23) leads to,

$$\bar{Q} = (\theta - 1) \frac{\gamma A \bar{p}_1 \bar{u}_1}{\gamma - 1}. \quad (26)$$

Introducing the relation for the heat release fluctuations as a function of the upstream velocity fluctuations in the linearised equations, the system can be expressed as

$$\mathbf{x}'_2 = \left(\begin{bmatrix} 1 & -\frac{\gamma \bar{p}_1 \bar{u}_1 (\theta - 1)}{c_1^2} & 0 \\ -\frac{(\theta - 1) \bar{u}_1}{\bar{p}_1} & 1 & 0 \\ \frac{c_1^2 (\theta - 1)}{\gamma (\gamma - 1) \bar{p}_1} & -\frac{(\theta - 1) c_1^2}{\gamma (\gamma - 1) \bar{u}_1} & \theta \end{bmatrix} + \begin{bmatrix} 0 & -\frac{\gamma \bar{p}_1 \bar{u}_1 (\theta - 1)}{c_1^2} A \text{FTF} & 0 \\ 0 & (\theta - 1) A \text{FTF} & 0 \\ 0 & \frac{c_1^2 (\theta - 1) A \text{FTF}}{(\gamma - 1) \gamma \bar{u}_1} & 0 \end{bmatrix} \right) \mathbf{x}'_1. \quad (27)$$

A peculiarity arises when the mean flow is zero, the energy equation gives the relation for the velocity fluctuations $u'_2 = u'_1$ and the mass flow $\bar{\rho}_1 u'_1 = \bar{\rho}_2 u'_2$ (eq. 16 and eq 17). Clearly these requirements on the fluctuations are conflicting when $\bar{\rho}_1 \neq \bar{\rho}_2$. This discrepancy has been discussed in the literature⁸ and it is shown that the unsteady mass flow rate is not necessarily conserved over the flame, as it depends on the density variations through the source. To resolve this problem the volume flow should be conserved rather than the mass flow.⁸ Then the following relationship between the upstream and downstream perturbations can be derived^{9, 10} yielding

$$\mathbf{x}'_2 = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & (\theta - 1) A \text{FTF} \end{bmatrix} \right) \mathbf{x}'_1. \quad (28)$$

IV. Discussion

IN the specific case of no flow, the coupling between the upstream and downstream fluctuations for the mass source eq. (21) and heat source eq. (28) have the same structure. If there is no coupling of the mass source to upstream pressure perturbations, $m_p = 0$, and the coupling constant between the upstream velocity fluctuations and the heat source m_u equals $\gamma^2 p_1 (\theta - 1) AFTF / c_1^2$, then the two sources can be considered to be equivalent.

In the more general case, when $\bar{u}_{1,2} \neq 0$, the structure of the solution is different. By taking the solution for the coupling constant of the mass source to obtain the same structure at zero flow,

$$m_p = 0, m_u = \frac{\gamma^2 p_1 (\theta - 1) AFTF}{c_1^2} \quad (29)$$

the two coupling matrices C_m and C_h in equations eq. (20) and eq. (27) are respectively given by,

$$C_m = \begin{bmatrix} 1 & -\frac{\gamma \bar{p}_1 \bar{u}_1 (\theta - 1)(\gamma + 1)}{c_1^2} AFTF & 0 \\ 0 & 1 + (\theta - 1) AFTF & 0 \\ 0 & \frac{(\theta - 1)}{\gamma \bar{u}_1} AFTF & 1 \end{bmatrix}, C_h = \begin{bmatrix} 1 & -\frac{\gamma \bar{p}_1 \bar{u}_1 (\theta - 1)}{c_1^2} (1 + AFTF) & 0 \\ \frac{-(\theta - 1) \bar{u}_1}{\bar{p}_1} & 1 + (\theta - 1) AFTF & 0 \\ \frac{c_1^2 (\theta - 1)}{\gamma (\gamma - 1) \bar{p}_1} & \frac{(\theta - 1) c^2}{\gamma \bar{u}_1 (\gamma - 1)} (1 + AFTF) & \theta \end{bmatrix}, \quad (30)$$

and we can compare the two solutions. In the case of the heat source, the downstream pressure and velocity fluctuations are a function of both the upstream velocity and pressure fluctuations because of the jump in mean temperature, this type of coupling is not present for the mass source. The strength of the velocity-pressure and pressure-velocity coupling is dependent on the velocity and it becomes larger at larger velocities.

The coupling of the downstream velocity fluctuations to the upstream pressure fluctuations is small as the coupling scales with the $\rho_1 M_1 / c_1$. When the FTF has a large value, then the effect of the temperature jump could be neglected compared to the effect of the oscillating heat release rate on the pressure downstream of the flame. These observations lead to another possibility where the mass source could be approximately equivalent to a heat source with respect to the pressure and velocity fluctuations. These conditions are that the flame transfer function has a large value, $FTF \gg 1$ and the velocity fluctuations are larger than the pressure fluctuations $|u'_1 / p'_1| \gg \gamma M_1 / c_1 FTF$ at the source position so that the downstream velocity fluctuations are dominated by the downstream fluctuations. It should be noted that the condition to have a large value for the FTF is not trivial as it is in general a function of frequency, the flame configuration and fuel mixture,^{2,11,12} which could limit the possibilities to mimic the source term of a flame using an acoustic driver in the presence of flow.

The largest differences between the mass source and the heat source can be seen in the energy equation. For the heat source the downstream energy fluctuations are strongly coupled to the pressure fluctuations, but for the mass source, there is no coupling at all. Also, for the heat source the downstream energy fluctuations are more strongly coupled to the velocity fluctuations for the heat source than for the mass source. In the light of acoustic perturbations, which are isentropic, the effect of large energy fluctuations downstream of the source could be of no influence to the acoustic propagation, because for acoustic perturbations the energy fluctuations are directly coupled to the pressure fluctuations. Unfortunately, this analysis cannot give a clear answer whether or not the large energy fluctuations will have a strong influence on the acoustic disturbances as the main difference between the two kinds of fluctuations, the propagation speed,¹³ is not taken into account in this simplified analysis.

The incentive of this study is to possibly find other ways to investigate the stability of combustors. Besides the source terms another important aspect of thermo-acoustic instabilities is the form of the acoustic field within the combustor. The importance of the acoustic field can be appreciated by looking at the Rayleigh criterion,³

$$R = \frac{1}{T} \int \dot{Q}' p' dt, \quad (31)$$

which shows that if R is larger than one, i.e. the heat release fluctuations are in phase with the pressure oscillations, the heat source is able to increase the energy of the acoustic field.

The sound field in the combustor is governed by the wave propagation speed within the various parts of the combustor and the boundary conditions to the acoustic field, such as the acoustic conditions at the inlet(s) and outlet(s) of the combustor. Downstream of a flame the temperature and the flow velocity of the gas is higher due to the steady heat release rate of the source. Acoustically, the effect leads to a discontinuity of the wave propagation speed across the heat source. Furthermore, the temperature of the gas and the increased flow speed also have an influence on the acoustic end conditions.¹⁴⁻¹⁶ Therefore, to be able to extrapolate stability results when using a mass source instead of a heat source in a combustor, these effects should be taken into account to obtain meaningful results.

V. Conclusion

The incentive of this study was to find an alternative way to investigate the stability of combustors using an electro-acoustic equivalent for the acoustic response of a flame. Therefore the similarities between a compact heat source and a compact mass source have been investigated using a methodology based on the Rankine-Hugoniot relations. The resulting set of equations have been linearised and both the sources have been considered to be linearly dependent on upstream acoustic perturbations. Then both systems of equations were reduced and compared with each other.

It has been shown, that for the zero flow case the two sources can be considered equivalent, but when there is a flow present, this equivalence is in general not longer maintained. It has been shown that under specific conditions, a similar coupling between the upstream and downstream perturbations for a mass source and heat source can be obtained when there is a mean flow present. These conditions are that the flame transfer function is large compared to unity and that the acoustic velocity perturbations are large compared to acoustic pressure perturbations at the source position.

Even though there is an acoustic equivalence between a mass source and a heat source under certain conditions, the presence of a steady temperature and flow speed jump due to the flame lead to a change of the acoustics within the system and these effects should be taken into account to be able to successfully predict combustion instabilities using a mass source.

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