

# Propagation and Generation of Acoustic and Entropy Waves Across a Moving Flame Front

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## Abstract

In analytical models of the propagation and generation of acoustic and entropy waves across a premixed flame, the relations that couple upstream and downstream flow variables often consider the flame as a discontinuity at rest. This work shows how the model of a flame at rest can misrepresent the generation of entropy waves, and how it leads to paradoxical results concerning the conservation of mass and volume flow rates across the flame. Such inconsistencies can be resolved by taking into account the movement of the flame in the coupling relations for flow perturbations. Analysis in a quasi-1D framework shows that in the absence of perturbations in equivalence ratio, the magnitude of the entropy waves generated across the flame are *first order* in Mach number and derive from interactions between the upstream acoustics and the *mean* heat release rate. For non-perfectly premixed flames, fluctuations in equivalence ratio may generate perturbations in entropy of *leading order* in Mach number. Furthermore, for the moving flame model conservation of volume flow rate across a passive, perfectly premixed flame appears as a natural consequence of mass and energy conservation.

*Keywords:* kinematic balance; premixed flame; acoustic scattering; entropy generation; volume consumption rate; heat release rate; mass flow conservation; Mach number

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## 1. Introduction

In lean combustion systems, one of the major challenges to technological progress is thermo-acoustic instability. Such instabilities may be caused by

fluctuations in pressure or velocity, i.e. acoustic disturbances, which impinge on the flame, causing the heat release rate to become unsteady. Fluctuations in heat release rate will in turn generate more acoustic disturbances, so that a feedback-loop is established, which may result in self-excited instability.

Acoustic perturbations at a flame can also cause so-called “entropy waves”, i.e. temperature inhomogeneities in the burnt gases that are transported convectively. As Marble and Candel [1] have explained, when such inhomogeneities experience acceleration downstream of the flame (e.g. through a nozzle), acoustic waves are generated in both upstream and downstream directions from the zone of acceleration. The upstream propagating component travels back into the combustion chamber, contributing to the acoustic oscillations in the system. This mechanism can also trigger thermo-acoustic instabilities, see [2–5].

According to Rayleigh [6], instabilities in a thermo-acoustic system can occur, when thermal and acoustic disturbances interact constructively. Therefore, understanding the mechanisms of acoustic and entropy waves generation across the flame, and their propagation in the system is crucial for prediction and control of the thermo-acoustic instabilities.

To predict system instabilities, the framework of low-order network models is widely employed [5, 7–13]. In this framework, a one-dimensional thermo-acoustic system is represented as a network of acoustic elements, each one characterized by its transfer matrix, which expresses the relations between the flow perturbations in velocity  $u'$ , pressure  $p'$  and entropy  $s'$  upstream of the element to the perturbations downstream, see [14].

In an idealized treatment, such relations may be derived analytically from the linearized conservation equations for mass, momentum and energy. The effect of a heat source on the acoustic field may also be deduced from these conservation equations. In this case, the analysis should describe the scattering of acoustic waves by the temperature and density gradients that result from mean heat release rate  $\bar{Q}$ , as well as account for the coupling between the fluctuations of heat release  $\dot{Q}'$  and the acoustic perturbations<sup>1</sup>.

The configuration considered in the present paper is depicted in Fig. 1. The heat source is regarded as a one-dimensional discontinuity. This is appropriate if the heat source is *compact*, i.e. if both acoustic and entropy

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<sup>1</sup>Here and in the following, overbars  $\bar{\cdot}$  denote mean values, while primed quantities  $\dots'$  refer to fluctuations around the mean.

wavelengths are much larger than the axial extent of the heat source. Moreover, it is often assumed that the heat source is *fixed at position*  $\bar{x}_f$ . A derivation of thermo-acoustic coupling relations by analysis of the conservation laws for mass, momentum and energy as they apply for a compact heat source at rest can be found in several prior studies, see e.g., [10, 15–17].

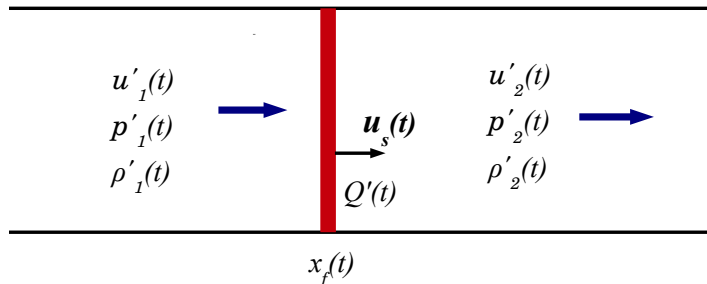


Figure 1: A compact heat source in quasi-1D flow with flow perturbations  $u', p', \rho'$  at upstream ('1') and downstream ('2') locations, respectively. For a heat source at rest, the velocity of the heat source in the laboratory frame  $u_s(t) = 0$ , thus the location of the source,  $\bar{x}_f$ , is constant in time.

However, paradoxical conclusions may result from the thermo-acoustic coupling relations for a heat source at rest. The first contradiction concerns the production of entropy waves by a heat source. The coupling relations for a heat source at rest imply that in general unsteady heat release  $\dot{Q}' \neq 0$  should result in the generation of entropy waves, i.e.  $s' \neq 0$  downstream of the heat source, see [10, 18, 19]. However, in the case of a perfectly premixed flame with homogeneous fuel/air premixture, the presence of significant entropy waves (i.e. temperature inhomogeneities) downstream of the combustion zone is difficult to justify physically, because in the case of adiabatic and complete combustion, the temperature increase across the flame and thus also the temperature downstream of the flame should be constant.

The second issue has been raised by Bauerheim et al. [17] for the case of a passive flame ( $\dot{Q}' = 0$ ) at rest, in the limit of vanishing mean flow Mach number. In the absence of mean flow, the energy conservation equation is reduced to conservation of volume flux, which implies that fluctuations of upstream ("1") and downstream ("2") velocities be equal,  $u'_1 = u'_2$ . However, this is in apparent contradiction with mass conservation, which for  $M = 0$

would seem to impose  $u'_1 \bar{\rho}_1 = u'_2 \bar{\rho}_2$ .

Bauerheim et al. [17] reexamined the quasi-1D conservation equations and observed that acoustic and entropy perturbations are coupled. At zero Mach number, a singularity in entropy is produced, which acts as an additional source term in the mass balance equation, which "explains why mass conservation of fluctuations is satisfied at non-zero Mach number while volume flow rate is conserved at zero Mach number" [17]. Thus the paradox is resolved, but the conclusions that result from the mathematical arguments are not easily reconciled with physical intuition.

For the two cases mentioned above, conclusions developed from linearized conservation equations for mass and energy are either apparently contradictory, or non-intuitive. This is rather unsatisfactory, since mathematical models should represent and clarify the actual physical problem. The physical meaning of the interdependency among entropy waves generation, unsteady heat release and mass flow conservation needs to be re-examined and contextualized by revisiting the coupling relations and the underlying assumptions.

In this work, it will be shown how the issues described can be resolved by relaxing the assumption that the heat source is at rest. Instead, the flame front will be considered as a moving discontinuity, which implies that movement of the heat source must be taken into account in the conservation equations. Equations which describe the propagation of small flow disturbances across a moving heat source were first derived by Chu [20, 21]. Although the moving flame model has been used since in many studies, see e.g. [7, 12, 22–29], its consequences on acoustic scattering and generation of entropy wave have not been fully explored. The present paper will analyze these consequences, by verifying the validity of the equations with physical arguments and examples.

In Section 2, we will introduce the difference between a moving heat source and a heat source at rest and explicate some consequences of movement of the heat source. In particular, the linearized conservation equations for perturbations of velocity, pressure and entropy across a moving heat source are analyzed (section 3). Section 4 turns to the particular case of a moving premixed flame front, with fluctuations in heat release rate, flame speed and flame surface area in response to upstream velocity perturbations. Next, the consequences of the flame front movement on entropy generation and acoustic scattering are examined for both perfectly and non-perfectly premixed flame (section 5-6). Finally, after terminology is established and the main results of this study are presented, section 7 discusses and contex-

tualizes previous publications on the model of a moving flame [7, 12, 22–29] and a flame at rest [10, 15–17, 30–32], respectively.

## 2. Motivation

In this section, we state the problems discussed in the previous section in mathematical terms and discuss some of the limitations that are implied with the application of the conservation equations of mass, momentum and energy to a heat source at rest. For the sake of simplicity, the case of a “passive source”, i.e. a heat source without fluctuations of the heat release rate,  $\dot{Q}' = 0$ , will be considered.

In presence of perturbations, relevant variables are divided into a mean component, which varies spatially, and a fluctuating component, which in general is a function of both time and space:

$$\varphi(x, t) = \bar{\varphi}(x) + \varphi'(x, t). \quad (1)$$

For analysis of the perturbations across a compact heat source, a commonly adopted approach is to consider the linearized conservation equations just upstream and downstream of a discontinuity. The equations for conservation of mass, momentum and energy read (c.f. [5, 10, 15–17, 32, 33]):

$$\begin{aligned} [\rho' \bar{u} + u' \bar{\rho}]_1^2 &= 0, \\ [p' + \rho' \bar{u}^2 + 2\bar{\rho} \bar{u} u']_1^2 &= 0, \\ [c_p \bar{T} (\rho' \bar{u} + u' \bar{\rho}) + \bar{\rho} \bar{u} (c_p T' + \bar{u} u')]_1^2 &= \dot{Q}'. \end{aligned} \quad (2)$$

Angular brackets  $[\varphi]_1^2$  with sub-/superscripts denote the difference between values of a flow variable  $\varphi$  upstream (“1”) and downstream (“2”) of the jump, i.e.  $[\varphi]_1^2 = \varphi_2 - \varphi_1$ . As mentioned in the previous section, the source region is considered as infinitesimally thin (i.e. compact with respect to acoustic and, in presence of mean flow, to entropy waves [10]) and fixed at position  $\bar{x}_f$  in the stream-wise direction (see Fig. 1). The discontinuity is regarded as a ‘black-box’, and its dynamic response to upstream perturbations is only represented by the source term  $\dot{Q}'$ .

In order to simplify the analysis, the fluctuating terms may be normalized by their respective mean values. Additionally, since flow regimes of interest are typically characterized by  $M \ll 1$ , terms of second or higher order in

Mach number may be neglected. Therefore, Eqs. (2) reduce to [12, 15, 27, 34]:

$$\begin{aligned}\frac{\rho'_2}{\bar{\rho}_2} + \frac{u'_2}{\bar{u}_2} &= \frac{\rho'_1}{\bar{\rho}_1} + \frac{u'_1}{\bar{u}_1}, \\ \frac{p'_2}{\bar{p}_2} &= \frac{p'_1}{\bar{p}_1} + \mathcal{O}(M^2), \\ \frac{u'_2}{\bar{u}_2} \left( \frac{\bar{T}_2}{\bar{T}_2 - \bar{T}_1} \right) &= \frac{\dot{Q}'}{\dot{Q}} + \frac{u'_1}{\bar{u}_1} \left( \frac{\bar{T}_1}{\bar{T}_2 - \bar{T}_1} \right) - \frac{p'_1}{\bar{p}_1} + \mathcal{O}(M^2).\end{aligned}\tag{3}$$

Considering the simplest case of a passive heat source ( $\dot{Q}' = 0$ ) and recalling the relation between entropy waves and fluctuations in pressure and density:

$$\frac{s'}{c_p} = \frac{p'}{\gamma \bar{p}} - \frac{\rho'}{\bar{\rho}},\tag{4}$$

Eqs. (3) give for the entropy waves produced downstream:

$$\frac{s'_2}{c_p} = \left( \frac{\bar{T}_1}{\bar{T}_2} - 1 \right) \left( \frac{u'_1}{\bar{u}_1} + \frac{p'_1}{\bar{p}_1} \right) + \frac{s'_1}{c_p} + \mathcal{O}(M^2).\tag{5}$$

Now, assuming that the upstream flow is isentropic  $s'_1 = 0$ , and considering that for linear perturbations and small Mach numbers pressure fluctuations are negligible compared to velocity fluctuations,

$$\frac{p'_1}{\bar{p}_1} = \mathcal{O}(\gamma M) \frac{u'_1}{\bar{u}_1} \ll \frac{u'_1}{\bar{u}_1},\tag{6}$$

it is reasonable to state that entropy production across the passive source is mainly due to the velocity perturbations:

$$\frac{s'_2}{c_p} \approx \left( \frac{\bar{T}_1}{\bar{T}_2} - 1 \right) \frac{u'_1}{\bar{u}_1}.\tag{7}$$

Such dependency is easily explained for a compact heater. In fact, in presence of constant total heat transfer rate ( $\dot{Q} = \dot{m} c_p \Delta T = \text{const}$ ), as the mass flow rate through the heater changes, the specific enthalpy jump will also change, generating temperature fluctuations. This is confirmed by Eq. (7), which shows that in response to a positive fluctuation in velocity  $u'_1$  (i.e. to an increase in mass flow), a drop in downstream specific entropy  $s'_2$  must take place.

However, for the case of a *perfectly premixed* flame, where the equivalence ratio is constant and cannot be affected by flow perturbations, the dependency expressed in Eq. (7) seems unphysical. In fact, in this case, the specific enthalpy jump is constant by definition and does not depend on the changes in mass flow rate.

Why do the coupling relations in Eqs. (2) and (3) fail to represent the physics of a perfectly premixed flame? The answer was given by B. T. Chu (see [21]), who pointed out that, acoustically speaking, the "chief difference between a flame and a heater" lies in the fact that the flame front has a dynamic behavior and can move from its mean position in response to upstream perturbations. Such movement, however, is not known a priori and is related to the local flame speed, "whereas for a heater, such movement is assumed to be given".

Indeed, the flame is not a fixed discontinuity, but can move, wrinkle or stretch in response to velocity perturbations. The presence of a moving discontinuity, instead of one at rest, impacts directly on the acoustic scattering and entropy generation across the heat source.

In fact, in presence of a moving discontinuity, the instantaneous mass flow crossing the flame front does no longer depend exclusively on the incoming flow conditions  $(\rho_1, u_1)$ , but also on the unsteady movement of the flame front itself. Considering a quasi-1D configuration – see Fig. 1 – the instantaneous mass flow crossing the discontinuity becomes:

$$\dot{m}(t) = (u_1(t) - u_s(t)) \rho_1(t), \quad (8)$$

where  $u_s(t)$  is the velocity of the flame front in the laboratory frame of reference, determined by kinematic balance between flame propagation and convection. This balance may be disturbed by flow perturbations, resulting in  $u'_s(t) \neq 0$ . Equation (8) shows that, depending on the response of the flame to acoustics in terms of  $u_s(t)$ , very different results for the mass conservation equation may be obtained.

Furthermore, mass-specific entropy  $s$  is – just like temperature  $T$  – an intensive quantity, related to the total heat release  $\dot{Q}(t)$  through the mass consumption at the flame front. Therefore, entropy and temperature fluctuations downstream of the flame can be correctly predicted *if and only if* the fluctuations in mass consumption are also quantified correctly.

The second issue mentioned in the introduction concerns the specific case of combustion in the limiting case of zero Mach number [17]. In absence of

mean flow,  $M = 0$ , the equations (2) for a passive flame  $\dot{Q}' = 0$  are simplified to:

$$\bar{\rho}_1 u'_1 = \bar{\rho}_2 u'_2, \quad (9)$$

$$p'_1 = p'_2, \quad (10)$$

$$\bar{T}_1 \bar{\rho}_1 u'_1 = \bar{T}_2 \bar{\rho}_2 u'_2. \quad (11)$$

At zero Mach, the combustion is perfectly isobaric,  $p_1 = p_2$ . With the ideal gas law,  $\rho_1 T_1 = \rho_2 T_2$ . It follows that the energy conservation Eq. (11) is reduced to

$$u'_1 = u'_2, \quad (12)$$

which may be interpreted as conservation of volume flow rate (per duct cross-sectional area) across the flame.

The validity of volume flow rate conservation at zero Mach is in agreement with the results of many thermoacoustic studies. It can be easily verified with the linearized Rankine-Hugoniot equations (see [2, 15, 34]):

$$\begin{aligned} u'_2 &= u'_1 + \bar{u}_1 \left( \frac{\bar{T}_2}{\bar{T}_1} - 1 \right) \left( \frac{\dot{Q}'}{\bar{Q}} - \frac{p'_1}{\bar{p}_1} \right) + \mathcal{O}(M^2), \\ p'_2 &= p'_1 + \mathcal{O}(M^2). \end{aligned} \quad (13)$$

With reference to Eq. (13), in absence of mean flow, the upstream and downstream velocity perturbations coincide, thus agreement with Eq. (12) is established.

However, the result in Eq. (12) does not agree with the mass conservation expressed by Eq. (9), which implies that the velocity fluctuations are amplified across the flame front as an effect of the temperature jump. Bauerheim et al. [17] proved the consistency of volume flow conservation at  $M = 0$  with mass flow conservation at  $M \neq 0$ . In their analysis, the entropy wave source terms were crucial to the solution of the problem. However, as mentioned previously, the generation of entropy waves downstream a premixed flame front should be a function of the enthalpy of the premixture, and, in principle, independent from mass flow conservation. Is there a solution to the inconsistency between volume and mass flow at  $M = 0$ , which does not depend on entropy generation, in the specific case of a premixed flame?

Although the question of mass vs. volume flow rate conservation might seem to be completely unrelated to the problem of generation of entropy



waves, we argue that its resolution can be found in the flame front movement as well. In fact, Eq. (9) expresses the continuity equation across a heat source at rest, while the shape and position of a flame front are usually not fixed, as mentioned previously (see Eq. (8)). Therefore, the mass conservation equation should be formulated for the case of a *moving* heat source. This formulation allows for the decoupling of entropy generation from the mass conservation equation, in the case of a moving flame front. Moreover, we shall demonstrate (see Chapter 6) that the condition expressed by Eq. (12) indeed implies for a passive flame that the flame front is moved back and forth by the velocity perturbations (see fig. 1), such that the velocity  $u'_s$  of the discontinuity itself equals the fluctuations in velocity  $u'_1$ , which in turn equal the downstream fluctuations  $u'_2$ .

The considerations presented for the two cases suggests that the flame front movement should be taken into account when analyzing the coupling of acoustic waves across a flame front and the generation of entropy waves by the flame. The next section will review the formalism introduced by B. T. Chu [20, 21] to analyze the perturbations across a moving premixed flame front. The analysis is extended up to 1st order in Mach.

### 3. Coupling relations for acoustics and entropy across a moving heat source

#### 3.1. Derivation

In gas dynamics, the Rankine-Hugoniot conditions express the relation between the flow variables upstream and downstream of an infinitesimally thin discontinuity, typically a shock wave. Across a moving shock wave that is propagating with velocity  $u_s$ , the Rankine-Hugoniot conditions for the vector  $\boldsymbol{\varphi}$  of flow variables are expressed as (see [35]):

$$u_s(\varphi_2 - \varphi_1) = f(\varphi)_2 - f(\varphi)_1. \quad (14)$$

In a system of conservation equations associated to the vector  $\boldsymbol{\varphi}$ ,  $\mathbf{f}(\boldsymbol{\varphi})$  is the vector of flux functions of the conserved variables.

In the present study the Rankine-Hugoniot conditions for a moving discontinuity will be applied to a plane, infinitesimally thin heat source in one dimensional flow, see see Fig. 1. For the moment, we make no further assumptions on the nature of the heat source. The application that we have

in mind is, of course, a compact premixed flame in a quasi-1D flow configuration. Pertinent consequences for the acoustic / entropy coupling relations will be explored in the next section.

The relevant governing equations are the non-homogeneous Euler equations:

$$\frac{\partial \boldsymbol{\varphi}}{\partial t} + \frac{\partial \mathbf{f}(\boldsymbol{\varphi})}{\partial x} = \mathbf{S}, \quad (15)$$

where  $\boldsymbol{\varphi}$ ,  $\mathbf{f}(\boldsymbol{\varphi})$  and the source term  $\mathbf{S}$  are of the form:

$$\boldsymbol{\varphi} = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \quad \mathbf{f}(\boldsymbol{\varphi}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E + p) \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 0 \\ 0 \\ \dot{q} \end{pmatrix}. \quad (16)$$

The conserved flow variables are mass, momentum and energy, while the inhomogeneity of the equations is due to the heat source term in the energy equation.

Applying the jump condition in Eq. (14) to the variables in Eq. (16), the conservation equations across a moving heat source (first presented by Chu [21]) read:

$$\rho_2 u_2 - \rho_1 u_1 = u_s(\rho_2 - \rho_1), \quad (17)$$

$$\rho_2 u_2^2 - \rho_1 u_1^2 + p_2 - p_1 = u_s(\rho_2 u_2 - \rho_1 u_1), \quad (18)$$

$$\rho_2 u_2 H_2 - \rho_1 u_1 H_1 - \dot{Q} = u_s(\rho_2 E_2 - \rho_1 E_1), \quad (19)$$

As already mentioned,  $u_s(t)$  is the velocity of the heat source in the laboratory frame of reference. As an extension of the original formulation in [35], which is frequently used in gas dynamics, here we have included a source term in the energy equation,  $\dot{Q}$ , which is the total heat release rate per cross-sectional area,  $\dot{Q} = \int_{\bar{x}_f^-}^{\bar{x}_f^+} \dot{q} dx$ . Recalling the relations between the total energy  $E$  and total enthalpy  $H$  for an ideal gas:

$$H = h + \frac{1}{2}|u|^2 = e + \frac{p}{\rho} + \frac{1}{2}|u|^2 = E + \frac{p}{\rho}, \quad (20)$$

$$\rho e = \frac{p}{\gamma - 1}, \quad (21)$$

and under the hypothesis of constant  $\gamma$  and  $c_p$ , it is possible to reformulate the energy conservation equation (19) in terms of primitive variables  $\rho$ ,  $u$

and  $p$ :

$$\begin{aligned} \frac{\gamma}{\gamma-1} (u_2 p_2 - u_1 p_1) + \frac{1}{2} (\rho_2 u_2^3 - \rho_1 u_1^3) - \dot{Q} = \\ = u_s \left( \frac{1}{\gamma-1} (p_2 - p_1) + \frac{1}{2} (\rho_2 u_2^2 - \rho_1 u_1^2) \right). \end{aligned} \quad (22)$$

Assuming flow in the low Mach regime, terms of second or higher order in Mach are neglected. This is appropriate for many combustion systems, as discussed below. The conservation equations reduce to (see [20],[27]):

$$[\rho u]_1^2 - u_s [\rho]_1^2 = 0, \quad (23)$$

$$[p]_1^2 = \mathcal{O}(M^2), \quad (24)$$

$$\frac{\gamma}{\gamma-1} [up]_1^2 = \dot{Q} + \mathcal{O}(M^2). \quad (25)$$

### 3.2. Linearized, non-dimensional conservation equations

In the present analysis, we consider a thin planar heat source oscillating around a fixed mean position  $\bar{x}_f$ , see Fig. 1. The decomposition for  $u_s(t)$  is thus:

$$u_s(t) = 0 + u'_s(t). \quad (26)$$

The equations for the mean quantities developed up to  $\mathcal{O}(M^2)$  are:

$$[\bar{\rho} \bar{u}]_1^2 = 0, \quad (27)$$

$$[\bar{p}]_1^2 = \mathcal{O}(M^2), \quad (28)$$

$$\frac{\gamma}{\gamma-1} [\bar{u} \bar{p}]_1^2 = \bar{\dot{Q}} + \mathcal{O}(M^2). \quad (29)$$

Subtracting Eqs. (27) - (29) from Eqs. (23) - (25), the conservation equations for perturbation quantities are obtained. Retaining only the fluctuations of first order and normalizing by the corresponding mean quantities yields for mass:

$$\frac{\rho'_2}{\bar{\rho}_2} - \frac{\rho'_1}{\bar{\rho}_1} + \frac{u'_2}{\bar{u}_2} - \frac{u'_1}{\bar{u}_1} = \frac{u'_s}{\bar{u}_1} \left( \frac{1}{\lambda} - 1 \right), \quad (30)$$

momentum:

$$\frac{p'_1}{\bar{p}_1} = \frac{p'_2}{\bar{p}_2} + \mathcal{O}(M^2), \quad (31)$$

and energy:

$$\frac{\dot{Q}'}{\dot{Q}} = \frac{p'_1}{\bar{p}_1} + \frac{u'_2}{\bar{u}_2} \left( \frac{\lambda}{\lambda - 1} \right) - \frac{u'_1}{\bar{u}_1} \left( \frac{1}{\lambda - 1} \right) + \mathcal{O}(M^2), \quad (32)$$

where  $\lambda \equiv \bar{T}_2/\bar{T}_1$  is the ratio between downstream and upstream mean temperatures. The normalized conservation equations were derived considering the normalized equation of state for perfect gases:

$$\frac{\rho'_2}{\bar{\rho}_2} + \frac{T'_2}{\bar{T}_2} = \frac{\rho'_1}{\bar{\rho}_1} + \frac{T'_1}{\bar{T}_1}. \quad (33)$$

Equations (30)-(32) are formulated for the case  $\bar{M}_1 \neq 0$ . In absence of mean flow, the linearized conservation equations cannot be expressed in the non-dimensional form  $(u'_2/\bar{u}_2, p'_2/\bar{p}_2, \rho'_2/\bar{\rho}_2)$ , but in terms of absolute fluctuating quantities  $(u'_2, p'_2, \rho'_2)$ .

### 3.3. Scattering and generation of acoustic and entropy waves

The equations (30) - (32) are a system of three equations in three unknowns. Two of them,  $u'_2$  and  $p'_2$ , are acoustic quantities, while the third,  $\rho'_2$ , is a function of both acoustic and entropy fluctuations, see Eq. (4). By substituting the density fluctuations it is possible to separate the acoustic scattering, i.e. the transmission and reflection of acoustic waves, from entropy generation. Omitting the algebraic manipulations and neglecting terms of  $2^{nd}$  order and higher order in Mach number, Eqs. (30) - (32) are re-written in terms of acoustic and entropy perturbations as follows:

$$\begin{pmatrix} \frac{u'_2}{\bar{u}_2} \\ \frac{p'_2}{\bar{p}_2} \\ \frac{s'_2}{c_p} \end{pmatrix} = \underbrace{\begin{bmatrix} \frac{1}{\lambda} & (\frac{1}{\lambda} - 1) & 0 \\ 0 & 1 & 0 \\ (\frac{1}{\lambda} - 1) & (\frac{1}{\lambda} - 1) & 1 \end{bmatrix}}_{\mathbf{M}} \begin{pmatrix} \frac{u'_1}{\bar{u}_1} \\ \frac{p'_1}{\bar{p}_1} \\ \frac{s'_1}{c_p} \end{pmatrix} + \underbrace{\begin{bmatrix} (1 - \frac{1}{\lambda}) \\ 0 \\ (1 - \frac{1}{\lambda}) \end{bmatrix}}_{\mathbf{N}} \frac{\dot{Q}'}{\dot{Q}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ (1 - \frac{1}{\lambda}) \end{bmatrix}}_{\mathbf{U}} \frac{u'_s}{\bar{u}_1} \quad (34)$$

The matrix  $\mathbf{M}$  and the vectors  $\mathbf{N}$  and  $\mathbf{U}$  express the respective influence of upstream perturbations, fluctuations of heat release rate and velocity of the heat source on the downstream perturbations.

The terms  $M_{11}$ ,  $M_{12}$ ,  $M_{21}$  and  $M_{22}$  represent the acoustic transfer matrix, since they relate exclusively acoustic fluctuations  $p'$ ,  $u'$  at upstream and downstream positions to each other. The matrix coefficients  $M_{21}$  and  $M_{22}$

show that the pressure does not exhibit a discontinuity at the heat source. This is a direct consequence of the low Mach number approximation, valid up to first order in Mach number ( $M$ ).

The matrix coefficients  $M_{31}$  and  $M_{32}$  will be non-zero if the mean heat release rate is non-zero, such that  $T_2 \neq T_1$  and thus  $\lambda \neq 1$ . In this case, entropy is coupled to the acoustics, i.e. upstream acoustic fluctuations can generate downstream entropy waves. Conversely, the scattering of acoustic waves is independent from upstream entropy perturbations, as  $M_{13} = M_{23} = 0$ .

The vector  $\mathbf{U}$  shows that the immediate influence of heat source movement is restricted to the production of entropy waves. Such a result is very interesting, since it confirms the idea expressed above (see Section 2) that flame front movement plays a fundamental role in the correct prediction of entropy generation across a premixed flame.

On the other hand, it appears that there is no direct influence of the velocity  $u'_s$  on the acoustic coupling relations (see  $U_1$  and  $U_2$ ). This suggests that the Rankine-Hugoniot relations (13) derived from the conservation equations for a generic heat source at rest (see e.g., [2, 5, 12, 15, 34]) are also valid for the case of a moving heat source, at least for perturbations up to first order in Mach number. However, before drawing conclusions on the *absolute* independence of the acoustic coupling relations from flame front movement, it is necessary to take into account interdependencies between fluctuations of the rate of heat release and the velocity of a heat source that result from its physical nature. Such interdependencies or constraints will be different for a heater, or a premixed flame, or a non-premixed flame, say, and need to be taken into account when trying to achieve closure for the system of Eqs. (34) by relating the heat release fluctuations to the upstream perturbations,  $\dot{Q}' = \dot{Q}'(p'_1, u'_1, s'_1)$ . This will be done in the next section for the case of a premixed flame in a quasi-1D flow configuration.

#### 4. Coupling relations for acoustics and entropy across a moving premixed flame

The thermoacoustic closure problem referred to at the end of the previous section is usually concerned with the quantification of heat release fluctuation  $\dot{Q}'$  in response to acoustic perturbations. This may be achieved, e.g., by determining a flame transfer function [12, 36–39]. In the present paper, we pursue a different goal and concentrate on the interdependencies between

flame speed, flame movement, flame surface area and heat release rate that result from the “physics” of the particular flame configuration at hand. This allows to describe and analyze the role of flame movement in acoustic scattering and entropy generation at a premixed flame. The analysis is carried out in a quasi-1D framework.

#### 4.1. Kinematic balance at a premixed flame front

For a ducted premixed flame one may define a specific (per duct cross-sectional area, units m/sec) rate of volume consumption,

$$\dot{V} \equiv \frac{1}{A_d} \int_{A_f} S_f dA = \frac{S_f A_f}{A_d}, \quad (35)$$

where  $A_d$  is the cross-sectional area of the duct, which is assumed constant,  $S_f$  the flame speed and  $A_f$  the flame surface area. The second identity holds if the flame speed  $S_f$  is constant along the surface of the flame, which is assumed in the following without essential loss of generality.

The following kinematic condition links the velocity  $u_1$  of premixture upstream of the flame with the velocity  $u_s$  of the flame in the laboratory frame and the volume consumption rate  $\dot{V}$ :

$$u_1(t) = \dot{V}(t) + u_s(t). \quad (36)$$

For an anchored flame in steady state, kinematic balance between flow and flame propagation implies (see also Fig. 1)

$$\bar{u}_1 = \bar{\dot{V}}, \quad (37)$$

with  $\bar{u}_s = 0$ . On the other hand, if the flow is perturbed, the flame will respond with changes in shape and position, such that

$$u'_1(t) = \dot{V}'(t) + u'_s(t). \quad (38)$$

Evidently, the volume consumption rate  $\dot{V}$  may be interpreted as an *effective flame speed*, akin to the *flame brush speed* in turbulent combustion modelling, which describes summarily the effect of flame shape on the consumption rate of premixture: elongated or wrinkled flames have a flame surface area  $A_f$  larger than the duct cross-sectional area  $A_d$  and thus consume more premixture than flat flames.

#### 4.2. Heat release and volume consumption for a premixed flame

In a premixed flame, heat release takes place when premixture is consumed at the flame front. It follows that the heat release rate for a lean premixed flame may be expressed in terms of the volume consumption rate  $\dot{V}$  as follows:

$$\dot{Q} = \rho_1 \dot{V} \phi q, \quad (39)$$

where  $\dot{Q}$  is the heat release rate per cross-sectional area of the duct (units  $W/m^2$ ),  $\phi$  the equivalence ratio of the premixture (which equals unity for stoichiometric conditions) and  $q$  the mass specific enthalpy of the premixture in stoichiometric conditions (units  $J/kg$ ).

For small perturbations, one formulates

$$\frac{\dot{Q}'}{\dot{Q}} = \frac{\rho_1'}{\rho_1} + \frac{\dot{V}'}{\dot{V}} + \frac{\phi'}{\phi}. \quad (40)$$

Combining expression (40) with the kinematic conditions Eqs. (36), (38) and Eq. (4), which relates perturbations in density to those of pressure and entropy, gives the following expression for the heat release fluctuations of a premixed flame:

$$\frac{\dot{Q}'}{\dot{Q}} = \frac{u_1'}{\bar{u}_1} - \frac{u_s'}{\bar{u}_1} + \frac{\phi'}{\phi} + \frac{1}{\gamma} \frac{p_1'}{\bar{p}_1} - \frac{s_1'}{c_p}. \quad (41)$$

How to interpret this result? It is obviously not an alternative formulation of the energy balance Eq. (32) across a compact heat source, as it does not involve any downstream quantities (index "2"). Also, it should not be interpreted as a flame transfer function. Premixed flames respond predominantly to perturbations of upstream flow velocity,  $\dot{Q}' \sim F_u(u')$ , or equivalence ratio,  $\dot{Q}' \sim F_\phi(\phi')$ . The corresponding flame transfer functions  $F_u$  and  $F_\phi$  (see Huber and Polifke [40]) depend in a non-trivial manner on detailed flow-flame interactions, which are not the subject of the present study. Instead, Eq. (41) should be regarded as an expression of interdependencies among flow perturbations, flame movement and rate of heat release. These constraints result from the physical effects that govern a lean, premixed flame front, see Eq. (36) and Eq. (39).

In the previous section, we have presented a general expression of the coupling relations for mass, momentum and energy across a compact, moving heat source, i.e. Eq. (34). The corresponding relations for the specific case

of a premixed flame are obtained by substituting the result (41) for relative fluctuations in heat release  $\dot{Q}'/\bar{Q}$  in Eq. (34), resulting in:

$$\begin{pmatrix} \frac{u'_2}{\bar{u}_2} \\ \frac{p'_2}{\bar{p}_2} \\ \frac{s'_2}{c_p} \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & \left(1 - \frac{1}{\gamma}\right) \left(\frac{1}{\lambda} - 1\right) & \frac{1}{\lambda} - 1 \\ 0 & 1 & 0 \\ 0 & \left(1 - \frac{1}{\gamma}\right) \left(\frac{1}{\lambda} - 1\right) & \frac{1}{\lambda} \end{bmatrix}}_{\mathbf{M}} \begin{pmatrix} \frac{u'_1}{\bar{u}_1} \\ \frac{p'_1}{\bar{p}_1} \\ \frac{s'_1}{c_p} \end{pmatrix} + \underbrace{\begin{bmatrix} \left(1 - \frac{1}{\lambda}\right) \\ 0 \\ \left(1 - \frac{1}{\lambda}\right) \end{bmatrix}}_{\mathbf{F}} \frac{\phi'}{\bar{\phi}} + \underbrace{\begin{bmatrix} \frac{1}{\lambda} - 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{U}} \frac{u'_s}{\bar{u}_1} \quad (42)$$

Here we have chosen to represent the interactions in terms of three groups of key variables: upstream flow perturbations  $u'_1, p'_1, s'_1$ , equivalence ratio perturbations  $\phi'$  and flame movement  $u'_s$ .

Note that upstream entropy fluctuations are often negligible,  $s'_1 \rightarrow 0$ . Also, it was shown in Section 2 that the pressure term  $p'_1/(\gamma\bar{p}_1)$  in Eq. (41) is of first order in Mach number. On the other hand, as mentioned above, there is no a priori justification for assuming that the order of magnitude of the flame front movement  $u'_s$  is small, since it depends in a non-trivial manner on the response of the flame to acoustic perturbations. Similarly, equivalence ratio perturbations are in general a leading order term in Eq. (41), as they result from fluctuations in air and fuel flow rates [12, 29]. It is worth noting that in the present quasi 1-D analysis we only consider fluctuations in time and not in space, and that therefore fuel inhomogeneities in the normal direction are not accounted for.

For the special case of a perfectly premixed flame, where fluctuations of equivalence ratio are absent, at low-Mach number and for isentropic upstream conditions, Eq. (41) reduces to:

$$\frac{\dot{Q}'}{\bar{Q}} = \frac{u'_1}{\bar{u}_1} - \frac{u'_s}{\bar{u}_1} + \mathcal{O}(M) \quad (43)$$

Correspondingly, the 2nd term on the r.h.s. of Eq. (42) drops out and we see that in this case the downstream flow perturbations  $(u'_2, p'_2, s'_2)$  are determined entirely by upstream flow perturbations  $(u'_1, p'_1, s'_1)$  and flame movement  $u'_s$ .

It should be noted that, even for  $\phi' = 0$ , the system described in Eq. (42) is still under-determined. In fact, the kinematic response of the flame front  $u'_s$  to upstream velocity perturbations is not known a priori. However, according to Eq. (43), such closure problem can be solved, with good approximation, by



quantifying the heat release rate response to upstream perturbations in velocity, i.e. by determining the flame transfer function  $F_u$  [40] of the perfectly premixed flame.

#### 4.3. Kinematic balance in time and frequency domains

Re-writing the kinematic balance Eq. (38) as

$$u'_s(t) = u'_1(t) - \dot{V}'(t) \quad (44)$$

it becomes evident that the position of the flame front is subject two competing effects: On the one hand, perturbations in the oncoming flow  $u'_1 > 0$  convect the flame front downstream. On the other hand, perturbations in volume consumption rate  $\dot{V}' > 0$  cause the flame front to propagate in the upstream direction.

This additional constraint was combined with Eq. (40) to derive Eq. (42) from (34) by elimination of the heat release  $\dot{Q}'$ . Alternatively, the equations may be written in terms of heat release  $\dot{Q}'$  instead of flame front velocity in the laboratory frame  $u'_s$ . Fundamentally, both descriptions are equivalent – but in the present context, the formulation that makes explicit use of the flame velocity  $u'_s$  is more convenient. Either way, the system of equations is not closed and a suitable closure model that relates either  $\dot{Q}'$  or  $u'_s$  to flow perturbations is needed.

Frequently, equations are transformed to the frequency domain and closure is achieved by relating the heat release  $\dot{Q}'$  to upstream velocity perturbations  $u'_1$  via a flame transfer function  $F_u(\omega)$  [38, 39]. If one formulates the governing equations with  $u'_s$  instead of  $\dot{Q}'$ , see Eq. (42), a “flame velocity transfer function”  $G_u(\omega)$  is required, such that

$$\frac{u'_s}{\bar{u}_1} = G_u(\omega) \frac{u'_1}{\bar{u}_1}. \quad (45)$$

Such a velocity transfer function  $G_u(\omega)$  allows to develop interesting conclusions on the relation between heat release rate and the movement of a perfectly premixed flame. Under the assumptions made in the previous section, the expression for the heat release rate Eq. (43) implies that

$$F_u(\omega) = 1 - G_u(\omega). \quad (46)$$

Consequently, at the low frequency limit  $\omega \rightarrow 0$ , where the transfer function for the heat release rate  $F_u$  of a perfectly premixed flame approaches

unity [41],  $G_u$  vanishes and the flame front must remain at rest. Indeed, in this quasi-stationary limit any non-zero flame front velocity  $u'_s \neq 0$  would imply flash back or extinction.

On the other hand, the typical low-pass character of premixed flames causes the gain of  $|F_u|$  to vanish at high frequencies, indicating that the motion of the flame front is dominated by rapid convective displacement by impinging velocity perturbations. The comparatively slow processes modulating the volume consumption rate, on the other hand, have negligible influence at high frequencies.

In the next two sections, we shall discuss further consequences of the interdependencies between flow and flame perturbations. In particular, the implications of Eqs. (41) and (43) for the generation of entropy waves and the scattering of acoustic waves at a premixed flame will be analyzed in detail.

## 5. Generation of entropy waves at a premixed flame

In the previous section, coupling relations for flow perturbations across a compact, moving premixed flame have been obtained. Consequences for the generation of entropy waves due to acoustic-flame interactions will be discussed in the following. Both perfectly as well as non-perfectly premixed flames will be considered. The analysis concentrates on the *generation* of entropy waves, thus it will be assumed in the following that there are no perturbations of entropy upstream of the flame ( $s'_1 = 0$ ).

In the literature on low-order modeling of flow-flame interactions it has been stated repeatedly (see e.g., [10, 18]) that the leading order term in downstream entropy fluctuation is different from zero whenever the normalized fluctuations in heat release rate do not equal the fluctuations in upstream velocity:

$$\frac{s'_2}{c_p} \neq 0 \quad \Leftrightarrow \quad \frac{\dot{Q}'}{\bar{Q}} \neq \frac{u'_1}{\bar{u}_1}. \quad (47)$$

This result indeed is inevitable if the flame is considered as a heat source at rest, its paradoxical consequences for the prediction of downstream entropy have been discussed in the Introduction.

On the other hand, according to Eq. (42), which describes a flame that may change its position in response to flow perturbations, downstream en-

entropy perturbations evaluate to

$$\frac{s'_2}{c_p} = \underbrace{\left(1 - \frac{1}{\lambda}\right) \frac{\phi'}{\bar{\phi}}}_{\mathcal{O}(1)} + \underbrace{\left(1 - \frac{1}{\gamma}\right) \left(\frac{1}{\lambda} - 1\right) \frac{p'_1}{\bar{p}_1}}_{\mathcal{O}(M)} + \underbrace{\frac{s'_1}{c_p}}_0. \quad (48)$$

The last term on the r.h.s. is zero by assumption (see above). The second term, which results from an interaction of acoustic perturbations with the mean temperature jump, is of first order in Mach number. A detailed analysis of this mechanism of entropy wave generation is given in the Appendix of this paper. The only term on the r.h.s. of Eq. (48) that is not negligible at vanishing Mach number is the first term, which describes the result of equivalence ratio fluctuations.

Eq. (48) may be written as follows:

$$s'_2 = c_p \left(1 - \frac{T_c}{T_h}\right) \frac{\phi'}{\bar{\phi}} + \mathcal{O}(M). \quad (49)$$

The interpretation of this result, which was already developed by Keller [34] and Polifke et al. [5], is rather straightforward: inhomogeneities in the fuel concentration of the premixture modulate the specific heat of combustion and thus result in fluctuations of downstream temperature and density.

For a *perfectly* premixed flame  $\phi' = 0$  and  $s'_2 = \mathcal{O}(M)$ , i.e. leading order entropy waves cannot be generated by this type of flame. This does not imply - as Eq. (47) for a flame at rest would suggest - that fluctuations of velocity cause a proportional and immediate perturbation of the heat release rate. Instead, referring to Eq. (40), we find that for a perfectly premixed moving flame

$$\frac{\dot{Q}'}{\dot{Q}} = \frac{\dot{V}'}{\dot{V}} + \mathcal{O}(M) \quad (50)$$

and conclude that fluctuations in heat release rate vary according to the kinematic response of the flame, which determines e.g. the shape and surface area of the flame and thus its volume consumption rate. However, perturbations of this kind do not generate entropy waves, as they only modulate the rate at which premixture is consumed by the flame, but they do not modulate the heat released per unit mass of premixture and thus do not generate - to leading order - inhomogeneities of entropy.

The analysis in this section has shown that for a premixed flame, the generation of entropy waves is neither a direct nor a necessary consequence

of unsteady heat release rate. Instead, entropy waves are generated to leading order in Mach number only from inhomogeneities in the equivalence ratio of the premixture. Additional entropy source terms, which are of first order in Mach number, result from the interaction of acoustic fluctuations with the mean heat release (see the Appendix). The contradiction discussed in the introduction of this paper is thereby resolved.

The problem of the correct estimation of entropy waves is strictly connected to the correct formulation of mass flow conservation equation. Indeed, across a perfectly premixed flame, no leading order temperature inhomogeneities can be produced, therefore  $\rho'_2/\bar{\rho}_2 \sim \mathcal{O}(M)$ . In the model of a *flame at rest* (Eqs. (3)), energy conservation gives for a passive source ( $\dot{Q}' = 0$ ):  $u'_1 = u'_2 + \mathcal{O}(M)$ , while, in absence of leading order entropy, mass flow conservation gives:  $u'_2 = (\bar{T}_2/\bar{T}_1)u'_1 + \mathcal{O}(M)$ . This result shows that inconsistencies indeed arise in the model of the flame at rest, when entropy generation is taken into account. The next section, on the other hand, will show how the *moving flame* model ( $u'_s \neq 0$ ) solves the inconsistency above at both zero and non-zero Mach number.

## 6. Coupling of velocity perturbations across a perfectly premixed flame

With reference to Eq. (42), we see that velocity perturbations downstream of a premixed flame may result from several contributions: upstream acoustic and entropy perturbations  $u'_1, p'_1, s'_1$ , flame movement  $u'_s$  and equivalence ratio fluctuations  $\phi'$ . The influences of  $\phi'$  and  $p'_1$  are easily explained: As discussed in the previous section, the fluctuations in equivalence ratio and pressure have, respectively, an impact of leading and first order on the downstream entropy fluctuations. A change in downstream entropy (thus, density) implies a modulation of the downstream velocity  $u'_2$ , due to mass conservation.

On the other hand, the influence of flame movement  $u'_s$  cannot be determined a priori in the general case. Equation (38) shows that flame movement  $u'_s$  will result from an imbalance between upstream flow velocity  $u'_1$  and specific volume consumption rate  $\dot{V}'$  of the flame. The latter depends – see Eq. (35) – on the response of flame area and flame speed to upstream flow perturbations. These effects are non-trivial, and require advanced analytical treatments and detailed experimental or numerical studies.

Nonetheless, it is possible to make simplistic assumptions on the flame response and explore the consequences, in order to highlight the role of the flame movement on the acoustic coupling across the flame. In this section, two limiting cases are analyzed: The first case concerns a passive, moving flame, which is convected around its mean position by velocity perturbations. The other case, on the contrary, is a flame at rest.

On both cases, perfect premixture ( $\phi' = 0$ ) will be considered, and we assume the upstream flow to be isentropic,  $s'_1 = 0$ . Only the consequences for downstream velocity  $u'_2$  will be discussed, since to leading order pressure and entropy are not influenced by the flame front movement in the case of a premixed flame (see Eq. (42)).

### 6.1. Limit Case I: Passive Moving Flame

According to Eq. (43), the interdependencies between flow perturbations, flame movement and heat release imply for a passive flame with  $\dot{Q}' = 0$  that

$$u'_1 = u'_s + \mathcal{O}(M). \quad (51)$$

The coupling relations across the flame – see the first line of Eq. (42) – imply in this case that

$$\frac{u'_2}{\bar{u}_2} = \frac{u'_1}{\bar{u}_1} \frac{1}{\lambda} + \mathcal{O}(M). \quad (52)$$

Due to mass conservation Eq. (17),  $\bar{u}_1 \lambda = \bar{u}_2$ , and we conclude that for a passive, perfectly premixed flame

$$u'_1 = u'_s + \mathcal{O}(M) = u'_2 + \mathcal{O}(M). \quad (53)$$

The physical interpretation of this result is very simple: the flame as a whole is convected back and forth by the velocity perturbations, such that flame movement  $u'_s$  corresponds perfectly to  $u'_1$  at every instant. There are – by assumption – no fluctuations of heat release rate,  $\dot{Q}' = 0$ , such that the condition  $u'_2 = u'_1$  appears obvious. In the limiting case of vanishing Mach number,  $M = 0$ , there is no transport of premixture towards the flame, such that heat release rate  $\dot{Q}$  and flame speed  $S_f$  must also vanish. This limiting case then corresponds to a temperature discontinuity, which is convected back and forth by the velocity perturbations.

The analysis of this limiting case of a *passive, moving* flame resolves the paradox of mass vs. volume flow rate conservation that was discussed in the Introduction and Section 2. It is gratifying to see that volume flow rate

conservation follows in a very straightforward manner from the conservation laws for mass, momentum and energy plus the kinematic conditions at the flame. The latter must be invoked, because it is not assumed that the flame is at rest.

### 6.2. Limit Case II: Flame at Rest

The second limit-case considers a flame at rest, such that  $u'_s = 0$  even in the presence of flow perturbation,  $u'_1 \neq 0$ . The interdependencies in Eq. (41) resulting from the physics of the flame require in this case that

$$\frac{\dot{Q}'}{\dot{Q}} = \frac{u'_1}{\bar{u}_1} + \mathcal{O}(M). \quad (54)$$

Furthermore, the coupling relations (42) across a premix flame demand that

$$\frac{u'_2}{\bar{u}_2} = \frac{u'_1}{\bar{u}_1} + \mathcal{O}(M), \quad (55)$$

which may be re-written as

$$u'_2 = \frac{\bar{T}_2}{\bar{T}_1} u'_1 + \mathcal{O}(M). \quad (56)$$

This result is in agreement with the linearized Rankine-Hugoniot relations (13) and corresponds to 'naive' application of the continuity equation to flow across a heat source: the velocity fluctuations downstream exceed those upstream as an effect of the temperature jump.

Returning to the physics of the problem, we realize with Eq. (38) that the flame can only be at rest if the specific volume consumption of the flame is always equal to the upstream velocity,  $\dot{V}' = u'_1$ . This condition results from the constraint that for a perfectly premixed flame, each fluid element passing through the flame experiences the same increase of temperature<sup>2</sup>. In other words, the flame would have to respond to a modulation of upstream velocity  $u'_1$  by adjusting the flame area or flame speed in such a manner, that the volume consumption rate is at each instant in time equal to the upstream velocity. This scenario might be a valid description of realistic

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<sup>2</sup>This constraint would be violated in the case of incomplete combustion or heat loss, but such effects are not considered here.

flame behavior in the limit of low frequencies. However, the response of a premix flame to flow perturbations is characterized by convective time lags. Thus one must conclude that in order to describe flame-acoustic coupling, the model of a heat source that is free to change its position in response to flow perturbations should be used in the general case ( $u'_s \neq 0$ ).

## 7. Discussion of previous works

Chu formulated the relations which describe the coupling of acoustic disturbances across a moving, compact flame front [20]. The present study adopted Chu's moving flame model to a quasi-1D framework, and explored consequences for coupling relations and generation of entropy waves. In the following, previous works related to these issues shall be re-visited, in order to further clarify the interdependencies among unsteady heat release rate, flame kinematic response, acoustics and entropy generation.

### 7.1. Survey of other moving flame front jump conditions

Blackshear [22] proposed matching conditions for velocity and density across a flame front that involve a "flame volume"  $V_f$ ,

$$A_D u_1(t) = A_f(t) S_f + \frac{\partial V_f(t)}{\partial t}, \quad (57)$$

$$A_D u_2(t) = A_f(t) S_f \frac{\rho_1}{\rho_2} + \frac{\partial V_f(t)}{\partial t}, \quad (58)$$

where  $A_D$  is the area of the duct and  $A_f$  the area of the flame front. The variation in time of the flame volume  $V_f(t)$  decouples the rate of incoming volume flow from the volume consumption rate of premixture at the flame front. This introduces an additional degree of freedom into the model of Blackshear, equivalent to the relaxation of the condition that the flame be at rest.

The flame front velocity in the laboratory flame  $u_s$  does not appear explicitly in Blackshear's analysis. But the relation to the ideas discussed in the present paper can be elucidated quite easily. Combining the above equations, one obtains:

$$\rho_2 u_2(t) - \rho_1 u_1(t) = \frac{(\rho_2 - \rho_1)}{A_D} \frac{\partial V_f(t)}{\partial t} \quad (59)$$

In steady state, the partial derivative in time in Eq. (59) is zero, incoming and outgoing volume flow and volume consumption at the flame front are

equal to each other. In the unsteady case, the rate of change of the flame volume  $\partial V_f(t)/\partial t = A_f(t)S_f - A_D u_1(t)$ , see Eq. (57). Substituting this result in Eq. (59) yields

$$\rho_2 u_2(t) - \rho_1 u_1(t) = (\rho_2 - \rho_1) \left( \frac{A_f(t)S_f}{A_D} - u_1(t) \right), \quad (60)$$

which should be interpreted as a combination of the mass conservation across a moving flame front Eq. (17) and the kinematic balance at the flame front Eq. (35).

Dowling [7] adopted the formalism of Blackshear [22] for the analytical study of a premixed flame anchored on a flame holder. Presuming a V-shaped flame front, it was possible to relate fluctuations in flame volume  $V_f$  to those of flame surface area  $A_f$  and heat release rate  $\dot{Q}$ . Compare this to the formalism developed in the present study, which does not presume a certain flame shape, but instead requires a transfer function for the heat release rate  $F_u$  or the flame velocity  $G_u$  to achieve closure, see Section 4.3. One may conclude that the formulations of Chu [20] and Blackshear [22] both share an important feature, i.e. they decouple the rates of incoming flow and volume consumption by introducing an additional degree of freedom, i.e. flame movement  $u_s$  and flame volume  $V_f$ , respectively. However, the conceptual differences between the two formulations resulted in further development of the models in different directions.

The matching conditions expressed by Eqs. (39) - (40) can be also found in the work by Merk [23, 24]. Note, however, that Merk considers only perfectly premixed flames, and mainly focused on the contributions of flame area (or volume) and upstream density.

Pelcé and Rochwerger [25, 26] adopted the moving flame model for the description of vibratory instability in cellular flames. Their analysis recovers the quasi 1-D formulation used in the present paper, the kinematic balance at the flame front (see Eqs. (6)-(9) in [26]) is given in the same form as in Eq. (38).

In the more recent literature on 1-D analysis of thermoacoustic systems, see e.g. [10, 15, 16], the moving flame model (i.e. the kinematic balance at the flame front) has not found extensive application. The reason lies perhaps in the fact that the flame front movement  $u'_s$  only appears in the mass conservation equation, while in the energy conservation equation it is found only in terms of higher order in Mach number (see Eq. (22)). Thus the differences between the energy conservation equation in the moving model and



the model of the heat source at rest are not readily apparent. As noted in Section 3, the Rankine-Hugoniot relations for  $u'$  and  $p'$  are exact for fluctuations up to first order in Mach and the flame kinematics can be accounted for implicitly in the fluctuating heat release rate  $\dot{Q}'$ . Inconsistencies between the two models arise as soon as entropy generation is taken into consideration.

### 7.2. Flame kinematics and unsteady heat release

Schuermans [12] in the analysis of a premixed, turbulent, swirl stabilized flame, proposed two different closures for a simplistic flame model, featuring only acoustic and equivalence ratio fluctuations. In one model the flame is at rest at an axial position (ref. Eq. (17) in [12]), while the other model regarded the flame as a discontinuity fluctuating around a mean position (ref. Eq. (30) in [12]). The system transfer matrix featuring the moving flame model gave a better agreement with experiments. The model of flame at rest can be recovered in our analysis from Eq. (40), assuming that  $u'_s = 0$  and  $\dot{V}' = u'_1$  in every instant. The moving flame model can be recovered from Eq. (40), considering that the flame speed and the flame area are both constant, i.e.  $u'_s = u'_1$  and  $\dot{V}' = 0$ . The unsteady heat release rate in the moving flame model is mainly due to equivalence ratio fluctuations. The flame area, however, was considered as constant in this work.

In his review on the modeling of premixed combustion, Lieuwen (see [28], Eq. (34)-(39)) made an explicit distinction between the heat release rate due to changes in flame area  $\dot{Q}'/\bar{Q}|_{u'}$  and the share due to changes in equivalence ratio  $\dot{Q}'/\bar{Q}|_{\phi'}$ . In the context of Lieuwen's analysis, the flame can be modeled as <sup>3</sup>:

$$\frac{\dot{Q}'}{\bar{Q}} = \frac{\dot{Q}'}{\bar{Q}}\Big|_{u'} + \frac{\dot{Q}'}{\bar{Q}}\Big|_{\phi'} = \frac{A'_f}{\bar{A}_f} + \frac{\int \Delta h'_R d\bar{A}_f}{\int \Delta \bar{h}_R d\bar{A}_f}, \quad (61)$$

where  $\Delta h_R$  is the heat of reaction, which is function of the equivalence ratio. Equation (61) is the same formulation as Eq. (40) up to leading order. The former quantity  $\dot{Q}'/\bar{Q}|_{u'}$  depends on the response of the flame surface  $A'_f/\bar{A}_f$  to upstream velocity fluctuations. In our analysis,  $\dot{Q}'/\bar{Q}|_{u'}$  corresponds to the change in total volume consumption rate  $\dot{V}'/\bar{V}$  in Eq. (36). On the other

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<sup>3</sup>the sensitivity of the local flame speed  $S_f$  to equivalence ratio fluctuations is not discussed here

hand,  $\dot{Q}'/\bar{Q}|_{\phi'}$  is related exclusively to the generation of the temperature inhomogeneities downstream. In the quantification of the downstream entropy waves, it is useful to separate the unsteady heat release rate associated to inhomogeneities in the premixture, from the unsteady heat release rate due to modulations in mass flow rate. Such distinction is useful because the area change only governs the amount of the premixture being burnt instantaneously, while the temperature jump is a function of the enthalpy of the premixture only. When determining generation of entropy waves by a premix flame, only the unsteady heat release per unit of mass should be taken into account. According to the formalism above, this corresponds to  $\dot{Q}'/\bar{Q}|_{\phi'}$ . Accounting explicitly for the flame front movement  $u'_s$  in the jump relations allows to decouple the entropy production  $s'_2$  from  $\dot{Q}'/\bar{Q}|_{u'}$ , as shown in Eq. (34) and (42). On the other hand, the model of a heat source at rest does not allow for such decoupling.

### 7.3. Models of entropy production by a compact heat source

Dowling [10, 31] analyzed the entropy production across a compact heat source at rest. Her analysis yields for the downstream entropy:

$$\frac{s'_2}{c_p} = \left(1 - \frac{1}{\lambda}\right) \left(\frac{\dot{Q}'}{\bar{Q}} - \frac{u'_1}{\bar{u}_1} - \frac{p'_1}{\bar{p}_1}\right), \quad (62)$$

where  $\dot{Q}(t)$  is the total heat release rate per unit of volume. Equation (62) shows that, in the case of a heat source at rest (e.g. a heated wire or a heat exchanger) entropy waves are determined predominantly by the unsteady heat release rate (or heat transfer rate) and upstream velocity fluctuations, given that pressure perturbations are of higher order in Mach number. Note that Dowling also distinguishes between the total heat release per unit of volume and per unit of mass, respectively, which is crucial for the correct prediction of entropy generation. These conclusions are in agreement with the analysis presented in Section 5.

However, the reader is cautioned that Eq. (62) is not applicable for perfectly premixed flames, unless  $\dot{Q}'/\bar{Q} = u'_1/\bar{u}_1$ , i.e. if  $|FTF| = 1$ , which is in general not the case. Eq. (50) may be satisfied in general only if the flame front to is allowed to move (i.e.  $u'_s \neq 0$ ).

Cumpsty [30] presented results in terms of downstream entropy production as a function of Mach number and non-dimensional frequency. The

kinematic balance at the flame front was not considered. However, Cumpsty made a reasonable prediction of the entropy waves, because he considered the mass-specific energy equation, instead of the total energy conservation equation. The entropy fluctuations downstream are formulated as:  $s'_2/c_p = q'/c_p \bar{T}_1$ , where  $q'$  corresponds to the mass-specific heat release. This treatment is in line with the arguments put forward in Section 5. Cumpsty's results for downstream entropy fluctuations in function of frequency are mainly constant, and not frequency-dependent, as Eq. (62) would suggest.

In the work of Polifke et al. [5] on the coupling between entropy and acoustic waves at a choked exit, only fluctuations in heat release rate that are due to changes in fuel mass fraction ( $\dot{Q}'_{Y_F}$ ) were accounted for in the determination of the strength of downstream entropy waves. Fluctuations in the momentary consumption rate due to coherent vortices contribute to perturbations in heat release rate, but do not contribute to the generation of entropy waves. Again, this treatment is in complete agreement with the arguments developed above.

Finally, it is noted that in case the flow through the heated region is not subject to a kinematic balance as would be the case heat exchangers or hot wires, the previous works on discontinuities at rest (Dowling [10]) and the considerations on the limit of vanishing Mach number by Nicoud and Wieczorek [32] and Bauerheim et al [17] represent a complete and solid analysis of the mechanisms of acoustic scattering and entropy generation.

## 8. Conclusions

In this paper, a detailed analysis of the conservation equations for acoustic and entropy perturbations across a moving heat source has been carried out. The analysis allows to develop a deeper understanding of the influence of the flame front movement on the acoustic scattering and entropy generation. At the same time, it resolves paradoxical conclusions that result from the assumption of a flame front at rest in a physically intuitive manner.

Important consequences on the substantial generation of entropy waves by a premixed flame were elucidated. Removing the hypothesis of the fixed position of the heat source and invoking instead kinematic balance at the flame front, it has been demonstrated that to leading order in Mach number temperature inhomogeneities downstream of a premixed flame are associated exclusively with inhomogeneities in the mass-specific heat of reaction of the

premixture, i.e. the fuel concentration. In the absence of equivalence ratio perturbations, and assuming complete combustion without significant heat loss due to convection or radiation, only a small amount of entropy fluctuations may be generated. These terms are of first order in Mach number, and result from interactions between upstream acoustics and the mean heat release rate.

In addition, the conservation laws for mass, momentum and energy at vanishing Mach number plus the kinematic matching conditions at a moving flame imply that the conservation of volume flow rate across a passive heat source follows in a very straightforward manner from mass conservation.

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## Appendix A. Analysis of terms of first order in entropy generation

In section 5 it was shown that to leading order entropy fluctuations downstream of a premixed flame are exclusively due to inhomogeneities in the equivalence ratio of the upstream premixture  $\phi'$ . In the absence of equivalence ratio fluctuations, entropy waves are of first order in Mach number,

$$\frac{s'_2}{c_p} = \underbrace{\left(1 - \frac{1}{\gamma}\right) \left(\frac{1}{\lambda} - 1\right)}_{\mathcal{O}(M)} \frac{p'_1}{\bar{p}_1}. \quad (\text{A.1})$$

The physical mechanism that leads to these terms is unclear. How are entropy waves generated in absence of inhomogeneities in the premixture? In this section, we will give a detailed explanation of such interaction.

The jump across the flame, at very low Mach numbers, can be regarded as an isobaric process at first order of approximation, see Eq. (31), with

mass-specific heat addition

$$q_{1 \rightarrow 2} = \int_1^2 dq = \int_1^2 T ds = \int_1^2 c_p dT = c_p(T_2 - T_1). \quad (\text{A.2})$$

The corresponding change in specific entropy equals

$$s_2 - s_1 = \int_1^2 ds = \int_1^2 c_p \frac{dT}{T} = c_p \log \frac{T_2}{T_1}. \quad (\text{A.3})$$

Combining these two results, which are an expression of the 1st and 2nd Law of Thermodynamics, one formulates for mean quantities

$$(\bar{s}_2 - \bar{s}_1) = \frac{\bar{q}}{\bar{T}_{ex}} \quad (\text{A.4})$$

where

$$\bar{T}_{ex} \equiv \frac{T_2 - T_1}{\log(T_2/T_1)} \quad (\text{A.5})$$

is the logarithmic mean temperature of the heat release process and  $\bar{q}$ , for a premixed flame, is equal to the specific heat of reaction of the premixture.

For a perfect premixture, the heat of reaction is constant, and so is the associated temperature jump  $T_2 - T_1$ . In presence of acoustics, however, the upstream pressure perturbation may cause a change in upstream temperature,  $p'_1 \rightarrow T'_1$ . As a consequence, the mean temperature of heat release  $\bar{T}_{ex}$  is also slightly altered.

$$(\bar{s}_2 - \bar{s}_1) + s' = \frac{\bar{q}}{\bar{T}_{ex} + T'_{ex}}. \quad (\text{A.6})$$

These qualitative considerations suggest that when upstream conditions  $(p'_1, T'_1)$  undergo small variations because of acoustic perturbations, a change in downstream entropy occurs, even in absence of unsteady heat release.

In order to determine the dependence of downstream entropy fluctuations on upstream acoustic perturbations, we consider isentropic upstream condition ( $s'_1 = 0$ ) and the linearized state equation for perfect gases, and reformulate Eq. (A.1) as:

$$\frac{s'_2}{c_p} = \left( \frac{1}{\lambda} - 1 \right) \frac{T'_1}{\bar{T}_1} = T'_1 \left( \frac{1}{\bar{T}_2} - \frac{1}{\bar{T}_1} \right), \quad (\text{A.7})$$

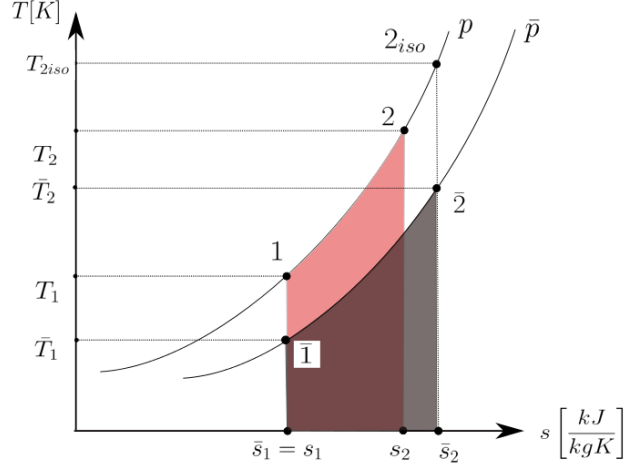


Figure A.2: The steady and unsteady heat transfer process represented in the T-s plane. It should be noted that the isobaric curves in the T-s plane are not drawn to scale, but serve only to give a qualitative description of the unsteady heat release process. In fact, in the real scale the two curves are much closer, therefore also the change in temperature ( $T_1$ ) is very small.

which shows that the entropy waves downstream are related to the temperature changes occurred upstream because of acoustic fluctuations. For an increase in upstream temperature, the associated entropy change is negative. This is directly explained by the 2nd Law of Thermodynamics, which states that, the higher the temperatures at which the heat addition occurs, the lower the increase in entropy.

The interaction between acoustics and mean heat release is well described in the T-s plane. In Fig. (A.2) two isobaric curves are represented in the T-s plane. On the lower curve, points  $\bar{1}$  and  $\bar{2}$  are the states of the gas before and after the combustion. The process takes place at mean pressure  $\bar{p}$ . The heat release per unit of mass is represented by the area under the curve ( $\bar{1}$ ,  $\bar{2}$ ,  $\bar{s}_2$ ,  $\bar{s}_1$ ). Starting from the initial state  $\bar{1}$ , a perturbation increases the pressure to  $p$ . Since the acoustic oscillations are isentropic (see [42]), the end state is at point 1, at temperature  $T_1$ . Given the constant heat release per unit of mass:

$$\bar{q} = \int_{\bar{s}_1}^{\bar{s}_2} T(s) ds|_{\bar{p}} = \int_{s_1}^{s_2} T(s) ds|_p \quad (\text{A.8})$$

the resultant of the combustion in presence of acoustic perturbations will

be at point 2. It is obvious that the entropy at point 2 is lower than the entropy at point  $\bar{2}$ . In order to suppress any entropy wave downstream, an increase in heat release rate would be required, such that the final state of the combustion is at point  $2_{iso}$ .

It is generally understood that the interaction between acoustics and a heat source produces entropy waves. Our analysis shows that in the case of a *perfectly* premixed flame the generation of entropy waves is only associated with upstream acoustics and *mean* heat release, but not fluctuations of heat release  $\dot{Q}'$ . Consequently, entropy waves downstream a perfectly premixed flame should not be identified with the fluctuations in the total heat release rate. Moreover, these fluctuations are first order in Mach and are negligible for  $M \ll 1$ . Of course, this argument assumes complete combustion as well as the absence of heat losses due to convective or radiative heat transfer. These results must be taken into account when discussing the role of entropy waves in thermoacoustic instability or indirect combustion noise.

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