



# ESTIMATING THE UNCERTAINTY IN STEPPED SINE MEASUREMENTS PERFORMED UNDER PARTIALLY STOCHASTIC CONDITIONS

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Stepped sine measurements are often performed in environments where there is a large contribution of background noise to increase the signal to noise ratio and obtain more accurate measurements. Due to time constraints or to guarantee the stability of the investigated system a single long measurement is often taken and the statistical properties of the results are based on this single measurement. When the background noise is not completely stochastic in nature, for example there is a tonal component present, the obtained statistics can lead to the wrong results because the underlying assumptions to derive these statistics are violated.

In this paper an expression is derived to estimate the uncertainty in a stepped sine measurements based on the background noise spectrum. In this way an accurate estimate of the uncertainty can be obtained even when it is not possible to perform enough statistically independent measurements. The results are based on synchronous demodulation using the Hilbert transform and the expressions are derived both in the continuous and discrete time domain so that they can be easily applied.

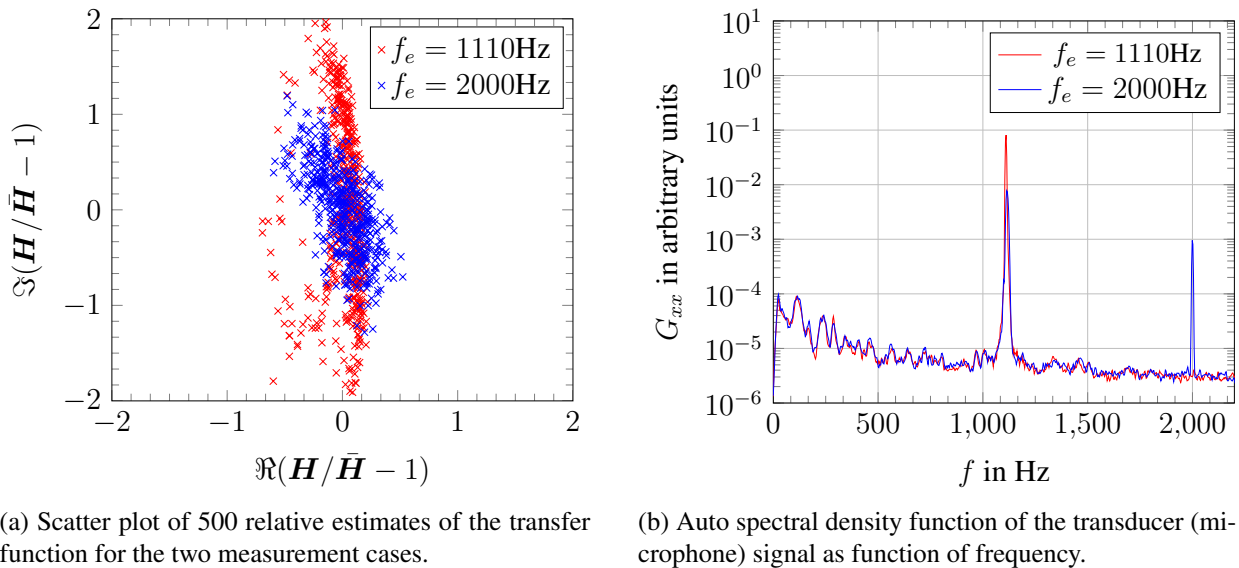
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## 1. Introduction

Stepped sine measurements are often performed in environments where there is a large contribution of background noise to increase the signal to noise ratio and obtain more accurate measurements. Due to time constraints or to guarantee the stability of the investigated system a single long measurement is often taken and the statistical properties of the final results are based on this single measurement. Unfortunately it is often impossible to measure enough independent sequences because the measurement times become too long, jeopardizing the time-invariant behaviour of the system.

In a single sine excitation measurement, the excitation signal is time-invariant and deterministic. Furthermore measurements are often performed in environments where tonal noise components are present. Therefore the statistical properties derived from the tools in signal processing based on non-deterministic and time-variant signals such as the coherence function become difficult to correctly interpret.

As an example, in Fig. 1 the real and imaginary parts of the estimated transfer function,  $H$ , between a microphone placed in a duct with flow and an excitation signal are plotted for two measurement frequencies  $f_e=1110$  Hz and  $f_e=2000$  Hz. The measurement-time for both frequencies is 10 seconds. In these measurements the error can be assumed to be on the output of the transducer



**Figure 1.** Scatterplot of the relative estimated transfer function between the source and transducer (left) and the auto spectral density at the transducer (right) for two stepped sine measurements,  $f_e$  equal to 1110 Hz (red) and 2000 Hz (blue) in an environment where a tonal component is present. Measurement time for both cases is 10 seconds.

alone because the excitation signal is affected only by electronic noise which is small compared to the excitation amplitude. The measurements contain a flow induced tonal noise in the region of 1050 - 1250 Hz as seen in the auto spectral density shown in Fig. 1. Estimates of the transfer function are calculated using the techniques described in [1]. Each instance of the estimate for both frequencies are plotted in a scatter plot depicted in Fig. 1. From the figure it can be immediately be seen that the distribution for both measurements is not circular and that the size of the scatter in the real and imaginary part of the transfer functions are not equal. These two observations are in disagreement with the commonly made assumptions and theoretical result that the error in the real and imaginary part of the transfer function are uncorrelated and of equal size [1, 3]. The disagreement is a result of the fact that the estimates are not statistically independent because of the measurement of only one long time series and not completely stochastic due to the presence of a tonal component in the uncorrelated part of the signal.

In this study the variances of a transfer function as function of the noise levels are derived which allows to estimate the variance using a separate measurement of the noise levels. This gives an estimate of true variance of the measurement when one would measure statistically independent instances. The relation is derived using a framework based on a synchronous demodulation technique [4].

First the concept of the synchronous demodulation technique is introduced and the thereafter a mathematical model to determine the expected value and variance of the determined complex pressures is presented. The analysis is performed in the time domain, and the properties of the (co)-variances of the real and imaginary parts of the determined transfer functions follow naturally from the analysis. The analysis is performed both in the discrete time domain and the continuous time domain such that the effect of discretization can be estimated and the derived expression can be directly applied to measuring situations.

## 2. Synchronous demodulation

Synchronous demodulation is a technique to remove the influence of errors uncorrelated with the excitation signal [4]. The idea is that the reference signal consists of harmonic signal with a

slowly varying instantaneous amplitude and frequency. By multiplying the measurement signal with a reference signal that has two components that are orthogonal with each other, e.g. a real and imaginary part, the result will be the projection of the measurement signal on both components of the reference signal. This approach gives then the in-phase part and out of phase part of the measurement signal w.r.t. the reference signal.

To represent a (random) signal in the complex domain such that it has two orthogonal components, the signal has to be transformed to an so-called analytic signal  $\mathbf{X}(t)$  (the bold typeface denotes complex variables):

$$(1) \quad \mathbf{X}(t) = x(t) + i\tilde{x}(t)$$

where  $\tilde{x}(t)$  is the Hilbert transform (HT) of  $x(t)$ . As the analytic signal is complex, it can be represented in phasor notation:

$$(2) \quad \mathbf{X}(t) = A(t)e^{i\psi(t)}$$

This phasor notation gives the time dependent information of the measured signal,  $A(t)$  is the instantaneous amplitude (envelope) and  $\psi(t)$  is the instantaneous phase of the measured signal. The relation between the instantaneous phase and the instantaneous frequency is given by [4]

$$(3) \quad \omega(t) = \frac{d\psi}{dt}$$

To obtain the in-phase and out-of phase parts of the measurements signals w.r.t the reference  $r$  signal, the reference signal is represented in the complex domain by its analytical signal  $\mathbf{r}$ . The measurement signal  $m(t)$  is then multiplied by the scaled analytical signal to obtain the projection  $\mathbf{P}$ :

$$(4) \quad \mathbf{P}(t) = m(t) \cdot \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|}$$

The measurement signal  $m(t)$  can be written as a summation of a response signal  $s(t)$  due to the excitation and contributions from other uncorrelated sources  $n(t)$ . If the system under study is linear, the response signal will also be a harmonic which has the same frequency (instantaneous phase) as that of the excitation signal. The amplitude  $A_s(t)$  and phase  $\phi(t)$  relative to the excitation signal may be time dependent and thus the signal can be written as:

$$(5) \quad s(t) = \frac{1}{2}A_s(t) (e^{i(\psi(t)+\phi(t))} + e^{-i(\psi(t)+\phi(t))})$$

The projection  $\mathbf{P}(t)$  of the measurement signal on to the analytic signal of the reference can now be written as:

$$(6) \quad \mathbf{P}(t) = [s(t) + n(t)] \cdot \frac{\mathbf{r}(t)}{|\mathbf{r}(t)|} = \frac{1}{2}A_s(t)e^{-i\phi(t)} + \frac{1}{2}A_s(t)e^{i(2\psi(t)+\phi(t))} + n(t)e^{i\psi(t)}$$

where  $n(t)$  represents the part of the measurement signal that is uncorrelated with the excitation. As  $\psi(t)$  is the instantaneous phase of the reference signal, which equals to  $\psi(t) = \omega_r t$  for a reference signal with constant frequency (following from Eq. (3)), we can see that the projection of the measurement signal on to the reference signal gives rise to a fast oscillating part,  $e^{2i\omega_r t + i\phi(t)}$ , a slow oscillating part  $e^{-i\phi(t)}$  and a part due to the uncorrelated signal  $n(t)e^{i\psi(t)}$  contribution.

Normally the relation between the input and output is time-invariant and thus by taken the time average of the projection an estimate of the relation between the input and output is obtained, i.e.

$$(7) \quad \tilde{\mathbf{P}} = A_s e^{-i\phi} + \mathbf{C}_n$$

where  $\mathbf{C}_n$  is the error introduced by the noise term and fast oscillating component. As term  $\tilde{\mathbf{P}}$  gives the relation between the input and output it can be seen as a transfer function. The  $\tilde{\mathbf{P}}$  is related to the transfer function commonly used in the signal analysis in the frequency domain [1], for which the absolute value of the input spectrum is normalized to one. In the remainder of the paper the term  $\tilde{\mathbf{P}}$  will be referred to as the transfer function.

### 3. Variance

To determine the variance of the transfer function between the excitation signal and the system response, first the expected value of the transfer function has to be determined. For the analysis, it is assumed that transfer function between the input and output signal is time-invariant and that the input signal is a sine of constant frequency  $\omega_r$  and amplitude  $A_r$  containing no noise. Under these circumstances, the analytic signal can be written as  $\mathbf{r}(t) = A_r \sin(\omega_r t) + A_r i \cos(\omega_r t)$ . The variance is calculated both in the discrete time and in the continuous time domain so that the influence of the sample frequency can be investigated.

First we determine the average of the projection for a limit amount of time,  $\tau = N\Delta t$ , where  $N$  is the number of samples taken and  $\Delta t$  the time between each sample, to obtain an estimate of the transfer function:

$$(8) \quad \tilde{\mathbf{P}} = \lim_{N \rightarrow \infty} \frac{1}{\tau} \sum_{i=0}^N \mathbf{P}(t_i) \Delta t$$

Thereafter an ensemble average of these estimates is taken to obtain the expected value of the transfer function  $E[\tilde{\mathbf{P}}] = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{j=0}^M \tilde{\mathbf{P}}_j$ . As shown in Eq. (6), the projection can be seen as a summation of three components, a slowly oscillating component, a fast oscillating component and a component related to an unwanted signal in the measurement signal and thus the expectation of the projection can be written as  $E[\tilde{\mathbf{P}}] = E[\tilde{\mathbf{P}}_0] + E[\tilde{\mathbf{P}}_{2f}] + E[\tilde{\mathbf{P}}_n]$ . The first component is the expected value of the estimate of the slowly oscillating component which does not depend on the measurement time,  $\tilde{\mathbf{P}}_0$ , the estimate of the component of the fast oscillating component  $\tilde{\mathbf{P}}_{2f}$  and the contribution of the noise source part  $\tilde{\mathbf{P}}_n$ . The constant and fast oscillating component do not have a random quantity and thus their expected values are given by

$$(9) \quad E[\tilde{\mathbf{P}}_0] = \frac{1}{2} A_s e^{-i\phi}$$

$$(10) \quad E[\tilde{\mathbf{P}}_{2f}] = \lim_{N \rightarrow \infty} \frac{1}{\tau} \sum_{i=0}^N \frac{1}{2} A_s e^{i(2\omega_r t_i + \phi)} \Delta t$$

The noise term is given by,

$$(11) \quad E[\tilde{\mathbf{P}}_n] = E \left[ \lim_{N \rightarrow \infty} \frac{1}{\tau} \sum_{i=0}^N n(t_i) e^{i\omega_r t_i} \Delta t \right]$$

where  $n(t_i)$  is the noise model. The noise model used in this paper assumes that the noise can be represented as a Fourier series with an arbitrary power spectral density  $G_{nn}(\omega_k)$  and that each component has a random phase  $\theta_k$

$$(12) \quad n(t_i) = \frac{1}{2} \sum_{k=0}^{\infty} \sqrt{2G_{nn}(\omega_k) \Delta\omega} (e^{i\omega_k t_i + i\theta_k} + e^{-i\omega_k t_i - i\theta_k})$$

To evaluate Eq. (11), the probability density function for  $\theta_k$  should be known. We assume that the pdf is constant and that the value of  $\theta_k$  is bounded by  $[0, 2\pi)$  resulting in an expectation of the noise term equal to  $E[\tilde{\mathbf{P}}_n] = 0$ . To calculate the variances of the real part and imaginary parts and the co-variances between the real and imaginary parts, first the deviation from the expected value has to be calculated. The deviation from the mean value is given by  $\tilde{\mathbf{P}} - E(\tilde{\mathbf{P}})$  and can be simplified to  $\tilde{\mathbf{P}}_n$  because the constant and fast oscillating term have no random component, their estimate and expected value are equal and the expected value of the noise term is equal to zero and thus the variance on the transfer function estimate is solely determined by the variance of the noise term,  $E(\tilde{\mathbf{P}})^2 = E(\tilde{\mathbf{P}}_n)^2$ .

### 3.1 Discrete time domain

The estimate of the noise term in the discrete time domain is given by:

$$(13) \quad \widetilde{\mathbf{P}}_n = \frac{1}{\tau} \sum_{k=0}^{\infty} \sum_{i=0}^N \frac{1}{2} \sqrt{2G_{nn}(\omega_k) \Delta\omega} \left( e^{i(\omega_k^+ t_i + \theta_k)} + e^{i(\omega_k^- t_i - \theta_k)} \right) \Delta t$$

were  $\omega_k^+ = \omega_r + \omega_k$  and  $\omega_k^- = \omega_r - \omega_k$ . Taken the imaginary and real part of the above estimate, the variance of the real and imaginary and the covariance between the imaginary and real part can be calculated. From the above equation it can be seen that each estimate is a completely random quantity for each  $k$  but not for each  $i$ . The variance of the real part is calculated by evaluating  $E(\Re(\widetilde{\mathbf{P}}_n)^2)$  and the variance in the imaginary part is calculated using  $E(\Im(\widetilde{\mathbf{P}}_n)^2)$ . As each element  $k$  is completely random, the summation over  $k$  of the square can be simplified using  $[\sum a]^2 = \sum a^2$  because the cross terms are uncorrelated and thus their expected value will be equal to zero. Evaluating the summation and then applying the expectation operator gives rise to the variance for the real and imaginary part of the estimate:

$$(14) \quad E \left[ \Re \left( \widetilde{\mathbf{P}}_n \right)^2 \right] = \frac{\Delta t^2}{\tau^2} \sum_{k=0}^{\infty} \frac{1}{2} G_{nn}(\omega_k) \Delta\omega \cdot \sum_{i=0}^N \sum_{j=0}^N \left[ \frac{1}{2} \cos(\omega_k^+(t_i - t_j)) + \frac{1}{2} \cos(\omega_k^-(t_i - t_j)) + \cos(\omega_r(t_i + t_j)) \cos(\omega_k(t_i - t_j)) \right]$$

$$(15) \quad E \left[ \Im \left( \widetilde{\mathbf{P}}_n \right)^2 \right] = \frac{\Delta t^2}{\tau^2} \sum_{k=0}^{\infty} \frac{1}{2} G_{nn}(\omega_k) \Delta\omega \cdot \sum_{i=0}^N \sum_{j=0}^N \left[ \frac{1}{2} \cos(\omega_k^+(t_i - t_j)) + \frac{1}{2} \cos(\omega_k^-(t_i - t_j)) - \cos(\omega_r(t_i + t_j)) \cos(\omega_k(t_i - t_j)) \right]$$

The covariances  $E(\Re(\widetilde{\mathbf{P}}) \Im(\widetilde{\mathbf{P}}))$  and  $E(\Im(\widetilde{\mathbf{P}}) \Re(\widetilde{\mathbf{P}}))$  are zero because of the orthogonality property of the sin and cos functions when evaluating the expectation operator.

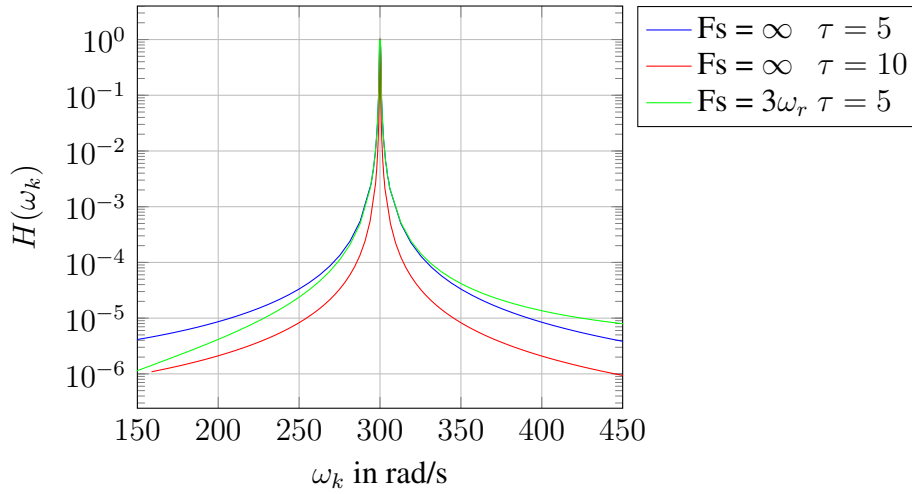
### 3.2 Continuous time domain

To obtain the expression for the continuous time-domain, the summation over time  $t_i$  is computed by taking the  $\lim_{N \rightarrow \infty}$  and evaluating the resulting integral.

$$(16) \quad \widetilde{\mathbf{P}}_n = \sum_0^{\infty} \frac{1}{2} \sqrt{2G_{nn}(\omega_k) \Delta\omega} \frac{1}{\tau} \left( \frac{e^{i(\omega_r + \omega_k)\tau} - 1}{i(\omega_r + \omega_k)} e^{i\theta_i} + \frac{e^{i(\omega_r - \omega_k)\tau} - 1}{i(\omega_r - \omega_k)} e^{-i\theta_i} \right)$$

The second term in the above equation has an fraction where the denominator could be zero, but the fraction is bounded for  $\omega_r \rightarrow \omega_k$ . In the same manner as in the discrete case, the variance of the real and imaginary part can be calculated for the continuous case

$$(17) \quad E \left[ \Re \left( \widetilde{\mathbf{P}} \right)^2 \right] = \sum_0^{\infty} \frac{1}{2} G_{nn}(\omega_k) \Delta\omega \cdot \left( \frac{1 - \cos \omega_+ \tau}{\omega_+^2 \tau^2} + \frac{1 - \cos \omega_- \tau}{\omega_-^2 \tau^2} - \frac{1 - 2 \cos \omega_- \tau - 2 \cos \omega_+ \tau + 2 \cos 2\omega_r \tau}{4\omega_+ \omega_- \tau^2} \right)$$



**Figure 2.** Calculated  $H(\omega_k)$  for different cases. Plotted points are such that  $\cos(\omega_-\tau) \approx 0$ . Excitation frequency at  $\omega_r = 300$  rad/s

$$(18) \quad \mathbb{E} \left[ \Im(\tilde{\mathbf{P}})^2 \right] = \sum_0^{\infty} \frac{1}{2} G_{nn}(\omega_k) \Delta\omega \cdot \left( \frac{1 - \cos \omega_+\tau}{\omega_+^2 \tau^2} + \frac{1 - \cos \omega_-\tau}{\omega_-^2 \tau^2} + \frac{1 - 2 \cos \omega_-\tau - 2 \cos \omega_+\tau + 2 \cos 2\omega_r\tau}{4\omega_+\omega_-\tau^2} \right)$$

with  $\omega_+ = \omega_r + \omega_k$  and  $\omega_- = \omega_r - \omega_k$ . Again for the continuous time case, the covariances between the real and imaginary part of the projection are equal to zero because of the orthogonality of the sin and cos functions when evaluating the expectation operator. As the summation is taken over all  $\omega$  and thus  $\omega_-$  can be equal to zero. The limits for  $\omega_- \rightarrow 0$  for the terms in the integrand exist and thus Eq. (17) and Eq. (18) are bounded and defined for all  $\omega_-$  except for  $\omega_- = 0$ .

## 4. Discussion

From both the discrete and continuous representations it can be seen that the variance in the determined projection is a function of both the noise spectrum itself and the acquisition parameters. Both the discrete and continuous variance can be represented as

$$(19) \quad \sigma^2 = \sum_0^{\infty} G_{nn}(\omega_k) \Delta\omega H(\omega_k)$$

using  $\sigma^2 = \mathbb{E} \left[ |\tilde{\mathbf{P}}_n|^2 \right] = \mathbb{E} \left[ \Re(\tilde{\mathbf{P}}_n)^2 \right] + \mathbb{E} \left[ \Im(\tilde{\mathbf{P}}_n)^2 \right]$ . Herein is  $H(\omega_k)$  the function describing how much of the noise at  $\omega_k$  is contributing to the variance as function of the acquisition parameters. In Fig. 2  $H(\omega_k)$  is depicted for the continuous case for two measurements times ( $\tau = 5$  seconds and 10 seconds). One solution of the discrete case is shown for a measurement time of 5 seconds and an acquisition rate corresponding to three times the excitation frequency.

The value of  $H$  close to the excitation frequency is given by the limit  $\omega_k \rightarrow \omega_r$  and equals one. The curves for the continuous case and discrete case overlap each other close to the excitation frequency and have deviation for frequencies far away from the excitation frequency. The deviation is small compared to the value close to  $H(\omega_r)$ , in the order of  $\mathcal{O}(10^{-6})$  and thus increasing the acquisition frequencies to very high values will not affect the variance of the end result significantly. This results in a way surprising as increasing the sample rate implies that more information is available to obtain a good estimate (under the assumption that each point is uncorrelated with each other). The only way

significant way to reduce the variance is by increasing the measurement time, which results in a more narrow peak in  $H$  at  $\omega_r$ . Using the continuous time model, the variance as function of the integration time  $\tau$  can be calculated and the variance of each estimate scales with  $\text{Var}(\widetilde{\mathbf{X}}) \propto \tau^{-1}$ . The variance of the mean of the estimates  $\text{var}(\overline{\mathbf{X}})$  can be related to the variance of each estimate when each estimate is statistically independent from each other showing that  $\text{var}(\overline{\mathbf{X}}) \propto \frac{1}{M\tau}$  and thus the variance of the determined transfer function is inversely proportional to the total measurement time  $T = M\tau$  and the signal to noise ratio is therefore proportional to  $\sqrt{T}$ .

In agreement with earlier results the imaginary part and real part are uncorrelated with each other. It is often assumed that the variance is circularly distributed [3, 2] in the complex domain, implying that the variance on the real and imaginary part are equal. From both the discrete and continuous results, Eqs. (14) - (18) it can be seen that this is not completely true. However, the last term that has a different sign is generally much smaller than the other two terms, and thus the error can be approximated to have a circular distribution.

## 5. Conclusion and Outlook

The paper investigates a way to determine the variance of a transfer function estimate under conditions where it is difficult to obtain a large number of statistically independent estimates. The variance of the transfer function estimate as function of the noise spectrum has been derived in the discrete and continuous time domain for a finite measuring time. It is shown that only the measurement time and the sampling frequency affect the variance and that the sampling frequency is of minor importance. The next step is to experimentally validate the derived relations.

## 6. Acknowledgments

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